Exercise 1. (5-5-5-5 points)
(1) Prove that every vector space $V$ over a field $F$ has a basis.
(2) Let $W$ be a subspace of $V$ and $S_0$ a basis of $W$. Prove that $V$ has a basis $S$ containing $S_0$. (Notice that $V$ is not necessarily of finite dimensions).
(3) Prove that every subspace $W$ of a vector space $V$ has a complement, that is, there is a subspace $U$ of $V$ such that every element $x \in V$ can be expressed in a unique way as $x = a + b$ where $a \in W$ and $b \in U$.
(4) Let $V$ be the vector space of all real-valued functions and $W$ its subspace of all odd functions. Find a complement of $W$. 
Exercise 2. (7-6-7)

(1) Find explicitly a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $T(1, 1, 0) = (1, 0)$ and $T(1, 0, 1) = (0, 1)$.

(2) Find the matrix representing $T$ in the standard bases $S_1, S_2$ of $\mathbb{R}^3$ and $\mathbb{R}^2$.

(3) Let $B_1 = \{(1, 1, 0), (1, 0, 1), (0, 0, 1)\}$ and $B_2 = \{(1, 1), (1, 2)\}$. Find the matrix representing $T$ in the bases $B_1$ and $B_2$. 
Exercise 3. (4-4-6-6 points)
Let $V$ be a vector space over a field $F$ and $T$ a linear operator on $V$.

(1) Prove that $\ker(T) \subseteq \ker(T^2)$ and $\text{range}(T^2) \subseteq \text{range}(T)$.

(2) Prove that if $T^2 = 0$, then $\text{range}(T) \subseteq \ker(T)$.

(3) Assume that $V$ is of finite dimension and $\text{range}(T^2) = \text{range}(T)$. Prove that $\ker(T) \cap \text{rang}(T) = \{0\}$.

(4) Find an example of an infinite dimensional vector space $V$ with a linear operator $T$ such that $\text{range}(T^2) = \text{range}(T)$ but $\ker(T) \cap \text{rang}(T) \neq \{0\}$. 
Exercise 4. (5-5-5-5)
Let $V$ be an $n$-dimensional vector space over a field $F$ and $T$ and $S$ two linear operators on $V$.

(1) Suppose that there are (ordered) bases $B_1$ and $B_2$ of $V$ such that $[T]_{B_1} = [S]_{B_2}$. Prove that there is an invertible operator $U$ on $V$ such that $S = UTU^{-1}$.

(2) Conversely, suppose that there is an inverse operator $U$ on $V$ such that $S = UTU^{-1}$. Prove that there are bases $B_1$ and $B_2$ of $V$ such that $[T]_{B_1} = [S]_{B_2}$.

(3) Application: Set $V = \mathbb{R}^2$, $T, S$ defined by $T(a, b) = (2a, 3b)$ and $S(a, b) = (3a, 2b)$.

(i) Find two ordered bases $B_1$ and $B_2$ of $V$ such that $[T]_{B_1} = [S]_{B_2}$.

(ii) Find an inverse operator $U$ on $V$ such that $S = UTU^{-1}$. 
Exercise 5. (5-5-5-5)

Let $F$ be a field and $V$ be a finite dimensional space over $F$.

(1) Find all linear transformations of $F$ as a vector space over itself.

(2) Let $f$ a linear functional on $V$ such that $W = \ker f$ is a hypersubspace of $V$.

Prove that for every linear functional $g$ of $V$ such that $g(W) = 0$, there is a scalar $c$ such that $g = cf$.

(3) Let $T$ be a linear operator on $V$ and $d$ a scalar in $F$ such that $Tu = du$ for some non-zero vector $u \in V$. Prove that there is a non-zero linear functional $h$ on $V$ such that $T^d(h) = dh$.

(4) Assume that $F = \mathbb{R}$, $V = \mathbb{R}^2$ and $T$ is defined by $T(a, b) = (2a, 0)$. Find $u$, $d$ and $h$ satisfying question (3).