King Fahd University of Petroleum and Minerals,
Department of Mathematics and Statistics- Term 141
Final Exam : Math 550, Linear Algebra
Duration: 3 Hours

NAME :

ID :
Exercise 1. (5-5-5)

Let $V = \mathbb{R}^3$ and let $B = \{(1, 0, 1), (1, 1, 1), (0, 0, 1)\}$ and $B' = \{(1, 0, 1), (0, 1, 0), (-1, 0, 0)\}$ two bases for $V$.

(1) Find the transition matrix $P$ from $B'$ to $B$.

(2) Let $T$ a linear operator on $V$ with $[T]_B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}$. Find $[T]_{B'}$.

(3) Is $T$ an isomorphism?
Exercise 2. (5-5-5-5 points)

Let $V$ be an $m$-dimensional vector space over $F$ and $T$ a linear operator with $m$ distinct characteristic values $c_1 = 1, c_2, \ldots, c_m$ such that $|c_j| < 1$ for $j = 2, \ldots, n$.

(1) Prove that for every vector $\alpha \in V$, $\lim T^n \alpha$ exists.

We define a linear operator $U$ on $V$ by $U \alpha = \lim T^n \alpha$.

(2) Find $\dim(\text{Nullspace}(U))$ and a basis $B_1$ for it.

(3) Find $\dim(\text{range}(U))$ and a basis $B_2$ for it.

(4) Find the matrix representing $U$ in the basis $B = B_1 \cup B_2$. 
Exercise 3. (4-5-5-6)
Let $V$ be an $n$-dimensional vector space over the field of rational numbers $\mathbb{Q}$ and $T$ a non-singular linear operator on $V$ such that $T^{-1} = T^2 + T$.

(1) Find the minimal polynomial of $T$.
(2) Prove that 3 divides $\text{dim} V$.
(3) Find the rational form matrix associated to $T$.
(4) Prove that every nonzero vector in $V$ is a cyclic vector if and only if $\text{dim} V = 3$. 
Exercise 4. (4-4-6-6 points)
Let $V$ be an $n$-dimensional vector space over a field $F$ and $T$ a linear operator.

(1) Assume that $T$ has exactly $n$ distinct characteristic values:
(i) Prove that every linear operator $U$ commutating with $T$ is diagonalizable.
(ii) Prove that if $U$ is a nilpotent operator commutating with $T$, then $U = 0$.

(2) Conversely, assume that $T$ commutes with no nonzero nilpotent operator.
(iii) Prove that if $c$ is a multiple characteristic value of $T$, then the Jordan block matrix $J_c$ in the Jordan matrix form of $T$ commutes with a nilpotent matrix $B_c$ to be determined.
(iv) Use the result of (iii) to prove that $T$ has exactly $n$ characteristic values.
Exercise 5. (5-5-5-5)
Let $V = \mathbb{R}^3$ endowed with its standard inner product and let $B$ the ordered basis $B = \{u_1 = (1, 0, 1), u_2 = (1, 1, 1), u_3 = (0, 0, 2)\}$.

(1) Apply Gram-Schmidt process to $B$ to obtain an orthonormal ordered basis $B' = \{v_1, v_2, v_3\}$ for $V$.

(2) Let $W = \text{span}\{v_2, v_3\}$ and $E$ the orthogonal projection of $V$ onto $W$. Find a formula for $E(x, y, z)$.

(3) Find $[E]_S$, the matrix representing $E$ is the standard basis $S$.

(4) Find $[E]_{B'}$. 
Exercise 6. (5-5-5-5-5)

Let $V$ be an $n$-dimensional complex inner product space and $T$ a normal linear operator.

(1) Prove that $\text{Nullspace}(T) = \text{Nullspace}(T^*)$.

(2) Prove that $\text{Nullspace}(T)$ is the orthogonal complement of $\text{range}(T)$ (that is, $\text{Nullspace}(T) = (\text{range}(T))^\perp$). Deduce that $\text{Nullspace}(T) = \text{Nullspace}(T^2)$.

(3) Suppose that there is two polynomials $f(X)$ and $g(X)$ relatively prime and $\alpha, \beta \in V$ such that $f(T)\alpha = g(T)\beta = 0$. Prove that $(\alpha | \beta) = 0$.

(4) Prove that there exist two linear self-adjoint operators $T_1$ and $T_2$ with $T_1 T_2 = T_2 T_1$ such that $T = T_1 + iT_2$ ($i$ is the complex number with $i^2 = -1$).

(5) Prove that there is a polynomial $h \in \mathbb{C}[X]$ such that $T^* = h(T)$.
Exercise 7. (5-5-5-5)

Let $V = \mathbb{R}^3$, $S = \{e_1, e_2, e_3\}$ its standard basis and $f$ the skew symmetric bilinear form on $V$ defined by $f(X, Y) = x_1y_2 - x_1y_3 - x_2y_1 + 2x_2y_3 + x_3y_1 - 2x_3y_2$.

(1) Find $[f]_S$.

(2) Find $\text{rank}(f)$.

(3) Let $W = \text{span}\{e_1, e_2\}$. Find a basis $B$ for $W^\perp$.

(4) Find $[f]_B$ where $B'$ is the basis $B' = \{e_1, e_2\} \cup B$. 