You are required to show your work on each problem on this exam. The following rules apply:

- If you use a “fundamental theorem” you must indicate this and explain why the theorem may be applied.

- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.
1. (10 points) Show that the set $C$, defined as

$$C := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 1 - x_1^2 - x_2^2 - x_3^2 \geq 0\},$$

is convex.
2. (20 points) Let $K$ be a nonempty convex cone in $\mathbb{R}^n$. Prove that $K = -K$ if and only if $K$ is a linear subspace of $\mathbb{R}^n$. 
3. (20 points) Let $\psi$ be a function defined on the interval $(a, b) \subset \mathbb{R}$; that is $\psi : (a, b) \to \mathbb{R}$.
Prove that $\psi$ is convex on $(a, b)$ if and only if

$$\psi(x) \leq \frac{x_3 - x_2}{x_3 - x_1} \psi(x_1) + \frac{x_2 - x_1}{x_3 - x_1} \psi(x_3),$$

for all $x_1, x_2, x_3$ such that $a < x_1 < x_2 < x_3 < b$. 

4. (20 points) Let $C$ be a convex subset of $\mathbb{R}^n$ and $f : C \to (-\infty, \infty)$ be a function defined on $C$. Define the function $\phi : [0, 1] \to (-\infty, \infty)$ as

$$\phi(t) = f(tx + (1 - t)y), \quad \text{for all} \quad x, y \in [0, 1].$$

Prove that

(a) the function $f$ is convex on $C$ if and only if the function $\phi$ is convex on $[0, 1]$,

(b) the derivative of the function $\phi$ is nondecreasing.
5. (20 points) Let $C$ and $D$ be two nonempty convex subsets of $\mathbb{R}^n$.
   (a) If $D$ is closed and $C \subset D$, prove that the recession cone of $D$ contains the recession cone of $C$.
   (b) Prove that the recession cones of $\text{ri}(C)$ and $\text{cl}(C)$ coincide.
6. (10 points) Consider the following optimization problem

$$
\begin{array}{ll}
\text{min} & f(x) \\
\text{subject to} & x \in X \\
\end{array}
$$

(P)

where $X \subset \mathbb{R}^n$ and $f$ is real-valued and differentiable function defined on an open set containing $X$. If $x^* \in X$ is a minimum of $f$, show that $y' \nabla f(x^*) \geq 0$ for every $y \in \mathbb{R}^n$ such that $x^* + \alpha y \in X$ for all $\alpha \in [0, \bar{\alpha}]$ for some $\bar{\alpha} > 0$. 