Math 580: Convex Analysis

Major Exam 2

Fall 2014

Time Limit: 120 Minutes

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a “fundamental theorem” you must indicate this** and explain why the theorem may be applied.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- If you need more space, use the back of the pages; clearly indicate when you have done this.

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Do not write in the table to the right.
1. (20 points) Let $C$ be a nonempty, convex subset of $\mathbb{R}^n$ and let $\bar{x}$ be a boundary point of $C$. Show that the sets $C$ and $\{\bar{x}\}$ form an extremal system.
2. (30 points) Let $C \subset \mathbb{R}^n$ be a convex set with $\bar{x} \in \mathbb{R}^n$.
   
   (a) Write a definition of the Normal Cone to $C$ at $\bar{x}$, $N(\bar{x}; C)$.

   (b) Prove that the normal cone to $C$ at $\bar{x}$ is the singleton $\{0\}$ whenever $\bar{x} \in \text{int}(C)$. 

3. (20 points) Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex function and $\bar{x} \in \text{dom} f$. Show only one of the following:

(a) $\partial^{\infty}(\bar{x}) = N(\bar{x}; \text{dom} f)$.
(b) $\partial(\bar{x}) = \{v \in \mathbb{R}^n : (v, -1) \in N((\bar{x}, f(\bar{x})); \text{epi} f)\}$. 
4. (30 points) Consider the following convex function

\[ f(x) = \begin{cases} 
  x^2 - 1, & \text{if } |x| \leq 1, \\
  \infty, & \text{otherwise on } \mathbb{R}.
\end{cases} \]

Find
(a) \( \partial^\infty f(0) \), \( \partial^\infty f(-1) \).
(b) \( \partial f(0) \), \( \partial f(-1) \).