Problem 1:

i. Let $\Omega_1$ and $\Omega_2$ be nonempty, convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively. For $(\vec{x}_1, \vec{x}_2) \in \Omega_1 \times \Omega_2$, show that

$$N((\vec{x}_1, \vec{x}_2); \Omega_1 \times \Omega_2) = N(\vec{x}_1; \Omega_1) \times N(\vec{x}_2; \Omega_2).$$

ii. Let $\Omega_1$ and $\Omega_2$ be convex subsets of $\mathbb{R}^n$ with $\vec{x}_i \in \Omega_i$ for $i = 1, 2$. Show that

$$N(\vec{x}_1 + \vec{x}_2; \Omega_1 + \Omega_2) = N(\vec{x}_1; \Omega_1) \cap N(\vec{x}_2; \Omega_2).$$

Problem 2:

Let $\Omega := \{x \in \mathbb{R}^n \mid x \leq 0\}$. Show that

$$N(\vec{x}; \Omega) = \{v \in \mathbb{R}^n \mid v \geq 0, \langle v, \vec{x} \rangle = 0\} \text{ for any } \vec{x} \in \Omega.$$

Problem 3:

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function differentiable at $\vec{x}$. Show that

$$N((\vec{x}, f(\vec{x})); \text{epi}(f)) = \left\{ \lambda \left( \nabla f(\vec{x}), -1 \right) \mid \lambda \geq 0 \right\}.$$ 

Problem 4:

Calculate the subdifferentials and the singular subdifferentials of the following convex function at every point of their domains:

(a) $f(x) = |x - 1| + |x - 2|, \quad x \in \mathbb{R}.$

(b) $f(x) = e^{|x|}, \quad x \in \mathbb{R}.$