Problem 1:

Let $\Omega_1$ and $\Omega_2$ be nonempty, convex subsets of $\mathbb{R}^n$. Suppose that $\text{ri} \Omega_1 \cap \text{ri} \Omega_2 \neq \emptyset$. Prove that

$$N(\bar{x}; \Omega_1 \cap \Omega_2) = N(\bar{x}; \Omega_1) + N(\bar{x}; \Omega_2) \text{ for all } \bar{x} \in \Omega_1 \cap \Omega_2.$$  

Problem 2:

Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ be convex functions satisfying the condition

$$\text{ri} \,(\text{dom} \, f_1) \cap \text{ri} \,(\text{dom} \, f_2) \neq \emptyset.$$  

Prove that for all $\bar{x} \in \text{dom} \, f_1 \cap \text{dom} \, f_2$ we have

$$\partial(f_1 + f_2)(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x}), \quad \partial^\infty(f_1 + f_2)(\bar{x}) = \partial^\infty f_1(\bar{x}) + \partial^\infty f_2(\bar{x}).$$  

Problem 3:

Find the Fenchel conjugate for each of the following functions on $\mathbb{R}$:

(i) $f(x) = e^x$.  
(ii) $f(x) = \begin{cases} -\ln(x) & x > 0, \\ \infty & x \leq 0. \end{cases}$  

Problem 4:

Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} -\sqrt{1 - x^2} & \text{if } |x| \leq 1, \\ \infty & \text{otherwise}. \end{cases}$$  

Show that $f$ is a convex function and calculate its directional derivatives $f'(-1; d)$ and $f'(1; d)$ in the direction $d = 1$.  