King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 695 - 1 & 6 Take home Final Exam
The First Semester of 2014-2015 (141)
Due date: December 28, 2014

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
P1, Q1		2
P1, Q2		1
P1, Q3		4
P1, Q4		1
P2, Q1		5
P2, Q2		5
Clarity and Presentation of your answers		2
Total		20

Problem 1:

Consider Problem (2.2) in [1]. This problem defines a semigroup $S_\epsilon(t)$ on $\mathcal{H}^0_\epsilon$. We admit that $S_\epsilon(t)$ has an absorbing set B_3 in $\mathcal{H}^2_\epsilon$. We also admit that the semigroup $S(t)$ of the unperturbed problem (cf. (2.1) in [1]) has an absorbing set B in $H_3 \times H_2$.

1. Show that there exists $c > 0$ independent of ϵ such that $\|S_\epsilon(1)z\|_{\mathcal{H}^1_1} \leq c$, for any $z \in B_3$.

2. Show that if $(u_0, w_0) \in B$ then the solution (ϕ, u) satisfies $\int_0^t \|u_{tt}\|^2 ds \leq c(t)$, $\forall t \geq 0$.

3. Show that

$$\|S_\epsilon(t)(u_0, w_0, u_1) - S_0(t)(u_0, w_0, \mathrm{L}(u_0, w_0))\|_{\mathcal{H}^0_1} \leq \sqrt{\epsilon}c(t),$$

for every $(u_0, w_0, u_1) \in \widetilde{B}_3 = S_\epsilon(1)B_3$.

4. Let $t_1 > 0$ such that $S_\epsilon(t)B_3 \subset B_3, \forall t \geq t_1$. We admit that $S_\epsilon(t) : [t_1, 2t_1] \times B_3 \to B_3$ is Hölder continuous. We also admit that $S_\epsilon(t)z_1 - S_\epsilon(t)z_2 = z^1(t) + z^2(t)$, and $z^i = (u^i, w^i, u^i_t)$, $i = 1, 2$, such that

$$\|z^1(t)\|_{\mathcal{H}^0_\epsilon} \leq ce^{-c't}\|z_1 - z_2\|_{\mathcal{H}^0_\epsilon} \tag{1}$$
$$\|z^1(t)\|_{\mathcal{H}^1_\epsilon} \leq ce^{c't}\|z_1 - z_2\|_{\mathcal{H}^0_\epsilon} \tag{2}$$

for any $z_i = (u_i, w_i, u_{1i}) \in \widetilde{B}_3$ and any $t \in [t_1, 2t_1]$.

Apply [1, Theorem 5.1] to deduce that, for every $\epsilon \in [0, 1]$, $S_\epsilon(t)$ has an exponential attractor $\mathcal{M}_\epsilon$ on $\widetilde{B}_3$, and that the family $\{\mathcal{M}_\epsilon\}_{\epsilon>0}$ is continuous at $\epsilon = 0$.

Problem 2:

Consider the singularly perturbed Cahn-Hilliard equation on $\Omega = (0, L)$ or $(0, L) \times (0, L)$

$$u_t - \Delta(\epsilon u_t - \Delta u + g(u)) = 0, \tag{3}$$
$$\partial_n u|_{\partial\Omega} = \partial_n \Delta u|_{\partial\Omega} = 0, \tag{4}$$

where $\epsilon \in [0, 1]$ and $g(u) = u^3 - u$.
When $\epsilon = 0$ the problem has an inertial manifold in the form

$$\mathcal{M} = \{p + \Phi(p),\ p \in PH_\alpha\},$$

for every $\alpha > 0$, where

$$H_\alpha = \{\psi \in L^2(\Omega),\ |\frac{1}{|\Omega|}\int_\Omega \psi dx| \leq \alpha\}$$

and P an orthogonal projection of finite rank.

1.) Show that Problem (3)-(4) has an inertial manifold in H_α of the form

$$\mathcal{M}_\epsilon = \{p + \Phi_\epsilon(p),\ p \in PH_\alpha\}$$

2. Show that there exists $C > 0$ such that, for every $\epsilon \in [0, 1]$,

$$\|\Phi_\epsilon(p) - \Phi(p)\| \leq C\epsilon,$$

for any $\|p\| \leq c$. Hint: borrow ideas of the proof of [2, Theorem 7.1].

References

[1] **A. Bonfoh**, *Dynamics of Hodgkin-Huxley systems revisited*, Applicable Analysis, 89 (2010), 1251-1269.

[2] **A. Bonfoh**, *The viscous Cahn-Hilliard equation with inertial term,* Nonlinear Analysis Series A: Theory, Methods and Applications 74 (3) (2011), 946-964.