King Fahd University Petroleum and Minerals
Department of Mathematics and Statistics

MASTER

MATH 201 - Term 142 - Exam II

MASTER

Duration: 120 minutes

KEY

Name:______________________ ID Number:______________________

Section Number:______________________ Serial Number:______________________

ClassTime: ______________________ Instructor's Name:______________________

General Instructions:

1. Calculators and Mobiles are not allowed.
2. This exam consists of two parts: Written and Multiple Choice.

Parts	Points	Maximum Points
Written		75
MCQ		25
Total		100

Part I: Written Questions

Instructions for Written Questions

1. This part has 7 written questions.
2. Answer the questions in the space provided.
3. Show your work. Points will be deducted for results without work.
4. Write clearly. Points may be deducted for poor presentation.
5. No credits will be given to wrong steps.

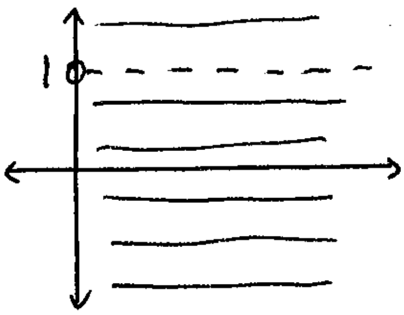
Question Number	Points	Maximum Points
1		14
2		10
3		7
4		14
5		8
6		10
7		12
Total		75

1. Let $f(x,y) = \dfrac{\sqrt{x}}{y-1}$.

(a) (5-points) Find and sketch the domain of f.

The domain of f is $\underbrace{\text{all } (x,y) \text{ in } \mathbb{R}\times\mathbb{R}}_{1\text{ pt}}$ so that $\underbrace{x \geq 0}_{1\text{ pt}}$ and $\underbrace{y \neq 1}_{1\text{ pt}}$

or, in mathematical notation Domain $= \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x \geq 0,\ y \neq 1\}$



region 1 pt

excluding $y=1$ 1 pt

(b) (2-points) Is the domain of f open, closed, or neither open nor closed? Explain.

Domain of f is neither open nor closed. 1 pt

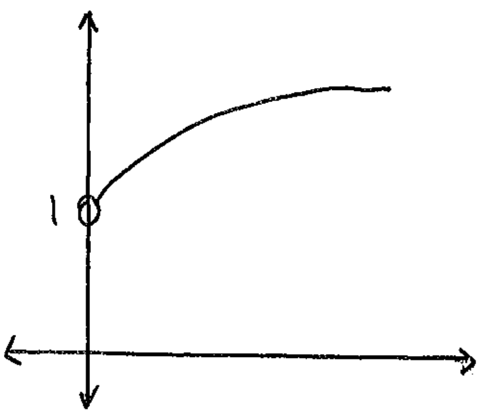
Because the boundary point $(0,1)$ is not included in the domain and the boundary points $(0,y)$ with $y \neq 1$ are in the domain 1 pt

(c) (7-points) Find an equation for and sketch the graph of the level curve of f that passes through the point $(4,3)$.

$\underbrace{f(4,3) = c}_{1\text{ pt}} \Rightarrow \dfrac{2}{3-2} = c \Rightarrow \underbrace{c = 1}_{1\text{ pt}}$

Equation of the level curve is $\dfrac{\sqrt{x}}{y-1} = 1$ 2 pts

After rewriting, the equation is $y = \sqrt{x} + 1$. So its graph is the graph of $y = \sqrt{x}$ shifted up by 1 units.



graph of $y = \sqrt{x}$ 1 pt

shift up 1 pt

excluding $(0,1)$ 1 pt

2. Let $f(x,y) = \dfrac{xy^2}{x^2+y^2}$.

a) (6-points) Determine the limit of f as $(x,y) \to (0,0)$.

For any (x,y) in $\mathbb{R}\times\mathbb{R}$, $y^2 \leq x^2+y^2$ 1 pt

This implies $\dfrac{x^2}{x^2+y^2} \leq 1$ 1 pt

From the latter, we obtain $\left|\dfrac{xy^2}{x^2+y^2}\right| = \dfrac{|x|\,y^2}{x^2+y^2} \leq |x|$ 1 pt

Then $0 \leq \dfrac{|x|\,y^2}{x^2+y^2} \leq |x|$ 1 pt

Since $\lim\limits_{(x,y)\to(0,0)} |x| = \lim\limits_{(x,y)\to(0,0)} 0 = 0$, by Sandwich Theorem

$\lim\limits_{(x,y)\to(0,0)} \dfrac{|x|\,y^2}{x^2+y^2} = 0.$ 1 pt

Hence, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2} = 0$ 1 pt

b) (2-points) Explain why f is not continuous at $(0,0)$.

f is not continuous at $(0,0)$ because $(0,0)$ is not in the domain of f.

c) (2-points) Define $f(0,0)$ in a way that extends f to be a continuous function at $(0,0)$. Justify your answer.

We define $\underbrace{f(0,0)=0}_{1\text{ pt}}$ because for continuity

we shall have $\underbrace{\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)}_{1\text{ pt}}$.

Alternative solution for 2a) using polar coordinates.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \underbrace{\lim_{r\to0}}_{\text{1 pt}} \underbrace{\frac{r\cos\theta\, r^2\sin^2\theta}{r^2}}_{\text{2 pts}} = \underbrace{\lim_{r\to0} r\sin^2\theta\cos\theta}_{\text{1 pt}} = \underbrace{0}_{\text{2 pts}}$$

3. (7-points) Find an equation of the plane through the points $P(1,2,-1)$, $Q(-1,0,3)$, and $R(0,3,-2)$.

$\overrightarrow{PQ} = \langle -2, -2, 4 \rangle$ (1 pt)

$\overrightarrow{PR} = \langle -1, 1, -1 \rangle$ (1 pt)

The vector $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ is normal to the plane. (2 pts)

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -2 & 4 \\ -1 & 1 & -1 \end{vmatrix} = \langle -2, -6, -4 \rangle$$ (1 pt)

Using the point $P(1,2,-1)$, we have.

$$-2(x-1) - 6(y-2) - 4(z+1) = 0$$ (1 pt)

$$2x + 6y + 4z = 10$$ (1 pt)

4. a) (6-points) If $f(x,y) = x^2 \sin(\pi y) + ye^{xy}$, then find $f_{xy}(2,-1)$.

$$f_x(x,y) = 2x\sin(\pi y) + y^2 e^{xy} \quad \text{2 pts}$$

$$f_{xy}(x,y) = 2\pi x\cos(\pi y) + 2y e^{xy} + xy^2 e^{xy} \quad \text{2 pts}$$

$$f_{xy}(2,-1) = 2\pi \cdot 2 \cdot \cos(-\pi) + 2(-1)e^{-2} + 2e^{-2} = -4\pi \quad \text{2 pts}$$

b) (8-points) The surface area of a closed right circular cylindrical container with radius r and height h is $S(r,h) = 2\pi r(r+h)$. If $r = 5cm$ is measured with an error at most 2% and $h = 15cm$ is measured with an error at most 4%, use differential to estimate the maximum percentage error in calculating the container's surface area.

Estimate of the percentage error $\approx \frac{dS}{S} \times 100$ 1 pt

$$dS = (4\pi r + 2\pi h)dr + 2\pi r\,dh \quad \text{2 pts}$$

$$\frac{dS}{S} = \frac{2r+h}{r+h}\frac{dr}{r} + \frac{h}{r+h}\frac{dh}{h} \quad \text{1 pt}$$

$$\frac{dr}{r} \leq 0.02 \ \text{1 pt} \quad \text{and} \quad \frac{dh}{h} \leq 0.04 \ \text{1 pt}$$

Then $\frac{dS}{S} \leq \frac{2.5+15}{20}\,0.02 + \frac{15}{20}\,0.04 = \frac{11}{200} = 0.055$ 1 pt

an estimate of the maximum percentage error is 5.5% 1 pt

5. (8-points) Find an equation for the plane tangent to the surface $\ln(xy - yz) = xz$ at the point $P(2, e^2, 1)$.

We consider the given surface as a level surface of

$$f(x,y,z) = \ln(xy - yz) - xz \quad (1\text{ pt})$$

$$f_x(x,y,z) = \frac{y}{xy - yz} - z \quad (1\text{ pt})$$

$$f_y(x,y,z) = \frac{x-z}{xy-yz} \quad (1\text{ pt})$$

$$f_z(x,y,z) = \frac{-y}{xy-yz} - x \quad (1\text{ pt})$$

$$\nabla f(x,y,z) = \left\langle \frac{y}{xy-yz} - z, \frac{x-z}{xy-yz}, \frac{-y}{xy-yz} - x \right\rangle \quad (1\text{ pt})$$

$$\nabla f(2, e^2, 1) = \left\langle 0, \frac{1}{e^2}, -3 \right\rangle \quad (1\text{ pt})$$

Then an equation of the tangent plane is

$$\left.\begin{array}{c} \frac{1}{e^2}(y - e^2) - 3(z-1) = 0 \\ \text{or} \\ \frac{y}{e^2} - 3z + 2 = 0 \end{array}\right\} \quad (2\text{ pts})$$

6. (10-points) Find the derivative of $f(x,y,z) = \sqrt{\frac{xy}{z}}$ at $P(3,-3,-4)$ in the direction of $\vec{u} = \langle 3,-1,4 \rangle$.

$|\vec{u}| = \sqrt{9+1+16} = \sqrt{26}$ 1 pt

$\vec{u}$ is not a unit vector. We will use the unit vector

$\vec{v} = \frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{26}} \langle 3,-1,4 \rangle$. 1pt

$f_x(x,y,z) = \frac{1}{2}\sqrt{\frac{y}{xz}}$ 1 pt

$f_y(x,y,z) = \frac{1}{2}\sqrt{\frac{x}{yz}}$ 1pt

$f_z(x,y,z) = -\frac{1}{2}\sqrt{\frac{xy}{z^3}}$ 1pt

$\nabla f(x,y,z) = \left\langle \frac{1}{2}\sqrt{\frac{y}{xz}}, \frac{1}{2}\sqrt{\frac{x}{yz}}, -\frac{1}{2}\sqrt{\frac{xy}{x^3}} \right\rangle$ 1pt

$\nabla f(3,-3,-4) = \left\langle \frac{1}{4}, \frac{1}{4}, -\frac{3}{16} \right\rangle$ 1pt

$Df_{\vec{v}}(3,-3,-4) = \nabla f(3,-3,-4) \cdot \vec{v}$ 2 pts

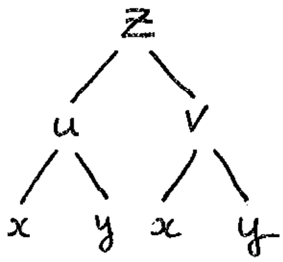
$= \frac{1}{\sqrt{26}} \left\langle \frac{1}{4}, \frac{1}{4}, -\frac{3}{16} \right\rangle \cdot \langle 3,-1,4 \rangle$

$= -\frac{1}{4\sqrt{26}}$ 1pt

7. (12-points) If $z = \tan^{-1}\left(\frac{u^2}{\sqrt{v}}\right)$ where $u = 2y - x$ and $v = 3x - y$, then find $\left.\frac{\partial z}{\partial y}\right]_{(x,y)=(2,2)}$.

$u(2,2) = 2$ 1 pt $\quad v(2,2) = 4$ 1 pt

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$ 2 pts



$$\left.\frac{\partial z}{\partial y}\right|_{(2,2)} = \left.\frac{\partial z}{\partial u}\right|_{(2,4)} \left.\frac{\partial u}{\partial y}\right|_{(2,2)} + \left.\frac{\partial z}{\partial v}\right|_{(2,4)} \left.\frac{\partial v}{\partial y}\right|_{(2,2)}$$ 1 pt

$$\frac{\partial z}{\partial u} = \frac{1}{1 + \frac{u^4}{v}} \cdot \frac{2u}{\sqrt{v}}$$ 1 pt $\Rightarrow \left.\frac{\partial z}{\partial u}\right|_{(2,4)} = \frac{2}{5}$ 1 pt

$$\frac{\partial z}{\partial v} = \frac{1}{1 + \frac{u^4}{v}} \cdot \frac{-u^2}{2v\sqrt{v}}$$ 1 pt $\Rightarrow \left.\frac{\partial z}{\partial v}\right|_{(2,4)} = \frac{-1}{20}$ 1 pt

$\frac{\partial u}{\partial y} = 2$ 1 pt $\quad \frac{\partial v}{\partial y} = -1$ 1 pt

Then $\left.\frac{\partial z}{\partial y}\right|_{(2,2)} = \frac{2}{5} \cdot 2 + \frac{-1}{20} \cdot (-1) = \frac{17}{20}$ 1 pt

END OF PART 1

Part II: Multiple Choice Questions

Instructions for Multiple Choice Questions

1. This part has 5 multiple choice questions.
2. Each question carries 5 points.
3. No partial credit.
4. Mark your answers on the table below.
5. No credit for answers not marked on the table below.

Question 1	A	B	C	D	E
Question 2	A	B	C	D	E
Question 3	A	B	C	D	E
Question 4	A	B	C	D	E
Question 5	A	B	C	D	E

1. The quadratic surface given by the equation $x^2 - 2x + y + z^2 - 4z + 5 = 0$ is

(A) an elliptical paraboloid

(B) an ellipsoid

(C) an elliptical cone

(D) a hyperboloid of two sheets

(E) a hyperbolic paraboloid

$x^2 - 2x + 1 + y + z^2 - 4z + 4 + 5 = 5$

$(x-1)^2 + y + (z-2)^2 = 0$

$-y = (x-1)^2 + (z-2)^2$

This is an elliptical paraboloid.

2. The distance from the point $S(5, 6, -1)$ to the line $L:\ x = 2 + 8t,\ y = 4 + 5t,\ z = -3 + 6t$ is equal to

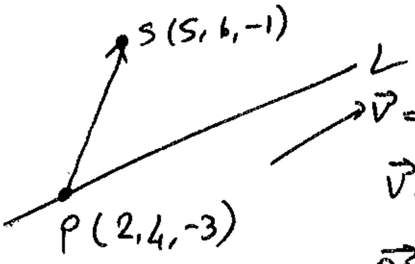
(A) $\dfrac{3}{5\sqrt{5}}$

(B) $\dfrac{1}{\sqrt{5}}$

(C) $\dfrac{3}{\sqrt{5}}$

(D) $\dfrac{2}{5\sqrt{5}}$

(E) $\dfrac{4}{5\sqrt{5}}$



$\vec{v}$ = direction vector of L.

$\vec{v} = \langle 8, 5, 6 \rangle$.

$\vec{PS} = \langle 3, 2, 2 \rangle$.

$$\vec{PS} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 8 & 5 & 6 \\ 3 & 2 & 2 \end{vmatrix} = \langle -2, 2, 1 \rangle \Rightarrow |\vec{PS} \times \vec{v}| = 3$$

$$|\vec{v}| = \sqrt{64 + 25 + 36} = \sqrt{125} = 5\sqrt{5}.$$

$$\text{distance} = \frac{3}{5\sqrt{5}}$$

3. The linearization $L(x,y)$ of $f(x,y) = y\ln\left(\frac{x}{y}\right)$ at $(e^2, 1)$ is equal to

(A) $L(x,y) = \frac{x}{e^2} + y$

(B) $L(x,y) = \frac{x}{e^2} + y + 2$

(C) $L(x,y) = \frac{x}{e^2} + 2$

(D) $L(x,y) = x + ye^2 + e^2$

(E) $L(x,y) = x - ye^2$

$f(e^2,1) = \ln(e^2) = 2$

$f_x(x,y) = \frac{y}{x} \Rightarrow f_x(e^2,1) = \frac{1}{e^2}$

$f_y(x,y) = \ln\left(\frac{x}{y}\right) - 1 \Rightarrow f_y(e^2,1) = 1$

$L(x,y) = 2 + (x-e^2)\cdot\frac{1}{e^2} + y - 1 = \frac{x}{e^2} + y$

4. If the equation $(y-1)z + e^{2z-1} = 2x^2y - \cos y + e$ defines z as function of two independent variables x and y, then $\left.\frac{\partial z}{\partial y}\right|_{(\sqrt{e},0,1)}$ is equal to

(A) 1

(B) -1

(C) $\frac{2e}{2e-1}$

(D) $\frac{2e+1}{2e-1}$

(E) $\frac{1}{2e}$

Consider $f(x,y,z) = (y-1)z + e^{2z-1} - 2x^2y + \cos y - e$.

$$\left.\frac{\partial z}{\partial y}\right|_{(\sqrt{e},0,1)} = -\frac{f_y(\sqrt{e},0,1)}{f_z(\sqrt{e},0,1)}$$

$f_y(x,y,z) = z - 2x^2 - \sin y \Rightarrow f_y(\sqrt{e},0,1) = 1 - 2e$

$f_z(x,y,z) = (y-1) + 2e^{2z-1} \Rightarrow f_z(\sqrt{e},0,1) = 2e - 1$

Then $\left.\frac{\partial z}{\partial y}\right|_{(\sqrt{e},0,1)} = 1.$

5. The normal line of the surface $4x^2 + y^2 + 2z = 6$ at the point $P(-1, 2, -1)$ is

(A) $x = -1 - 8t,\ y = 2 + 4t,\ z = -1 + 2t$

(B) $x = -1 + 8t,\ y = 2 + 2t,\ z = -1 + 2t$

(C) $x = -1 - 8t,\ y = 2 - 4t,\ z = -1 - 2t$

(D) $x = -1 - 4t,\ y = 2 + 4t,\ z = -1 + t$

(E) $x = -1 + 4t,\ y = 2 - 2t,\ z = -1 + t$

Let $f(x, y, z) = 4x^2 + y^2 + 2z - 6$.

$\nabla f(x, y, z) = \langle 8x, 2y, 2 \rangle \Rightarrow \nabla f(-1, 2, -1) = \langle -8, 4, 2 \rangle$

Using the point $P(-1, 2, -1)$

$x = -1 - 8t \qquad y = 2 + 4t \qquad z = -1 + 2t.$

END OF PART II.

DO NOT FORGET TO MARK YOUR ANSWERS ON THE TABLE PROVIDED AT THE BEGINNING OF PART II.