Name: ____________________________  ID#: __________________
Instructor: ______________________  Sec #: ______  Serial #: ______

Mobiles and calculators are not allowed in this exam.
Write all steps clear.

<table>
<thead>
<tr>
<th>Question #</th>
<th>Marks</th>
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<tr>
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Q: 1 (8+4 points) Consider the Sturm–Liouville problem
\[ y'' - \tan xy' + \lambda y = 0 \text{ with } y\left(\frac{x}{2}\right) = 0, \ y\left(\frac{x}{3}\right) = 0. \]

(a) Put the differential equation in self–adjoint form and write its weight function.

(b) If \( y_n \) and \( y_m \) are two eigenfunctions corresponding to two different eigenvalues, write the orthogonality relation.
Q:2 (22 points) Use separation of variables method to solve the problem

\[5 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \ t > 0,\]

subject to the boundary and initial conditions

\[u_x (0, t) = 0, \quad u_x (\pi, t) = 0, \quad t > 0,\]
\[u (x, 0) = x, \quad 0 < x < \pi.\]
Q:3 (22 points) Use Laplace transform to solve the problem

$$4 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x > 0, \quad t > 0,$$

subject to the boundary and initial conditions

$$u(0,t) = f(t), \quad \lim_{x \to \infty} u(x,t) = 0, \quad t > 0,$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0, \quad 0 < x < 1,$$

where

$$f(t) = \begin{cases} \sin \pi t & 0 < t \leq 1 \\ 0 & t > 1 \end{cases}$$
Q:4 (22 points) Use separation of variables method to solve the problem

\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < 2, \quad 0 < z < 5, \]

subject to the boundary conditions

\[
\begin{align*}
  u(2, z) &= 0, \quad 0 < z < 5 \\
  u(r, 0) &= 0, \quad 0 < r < 2 \\
  u(r, 5) &= 4, \quad 0 < r < 2
\end{align*}
\]

Also solution \( u(r, z) \) is bounded at \( r = 0 \).
Q:5 (22 points) Find the steady-state temperature $u(r, \theta)$ in a sphere of radius 2 by solving the problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0, \quad 0 < r < 2, \quad 0 < \theta < \pi,$$

subject to the boundary condition

$$u(2, \theta) = 1 + \cos(\theta), \quad 0 < \theta < \pi.$$
Q:6 (20 points) Find Fourier integral representation of

\[ f(x) = \begin{cases} 
0, & x < -1 \\
2, & -1 < x < 0 \\
-2, & 0 < x < 1 \\
0, & x > 1 
\end{cases} \]
Q:7 (20 points) Use appropriate Fourier transform to find the temperature $u(x, t)$ in a semi-infinite rod if $u_x(0, t) = 0$ and $u(x, 0) = e^{-x}$, $x > 0$. 