King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 531 (Real Analysis) **Term 142**

Exam 2 : April, 25, 2015 **Time allowed: 2hrs**

Question Number	Marks	Maximum Points
1		2
2		2
3		2
4		2
5		2
6		2
7		2
8		4
9		2
Total		20

(Q1) a) Define Lebesgue integral of a simple measurable function f.

b)Find Lebesgue integral of the following functions:

i) $f=$ characteristic function of irrationals

ii) $g(x) = 5$ for all $x \in C$ (Cantor set).

iii) $f_n : \mathbb{R} \to \{0, 1\}$ given by $f_n(x) = \chi_{[n,n+1)}$, (Q2) Let f be a bounded and measurable function defined on a set E of finite measure. Then prove that $\left|\int_E f\right| \leq \int_E |f|$.

(Q3) Let $g(x) = \begin{cases} 0 & x \in \left[0, \dfrac{1}{2}\right] \\ 1 & x \in \left(\frac{1}{2}, 1\right]. \end{cases}$

Define $f_{2n}x = g(x)$ and $f_{2n+1}(x) = g(1-x)$ for all $x \in [0,1]$. Check whether or not the conclusion of Fatou's Lemma holds for the sequence $\{f_n\}$.

(Q4) Let $\{f_n\}$ be a monotone increasing sequence of non-negative measurable functions defined on E and $f = \lim_n f_n$. Then $\int_E f = \lim_n \int_E f_n$ holds. Show by means of an example that this conclusion need not hold for a decreasing sequence $\{f_n\}$.

(Q5) If f and g are Lebesgue integrable on E, then show that $(f+g)$ is integrable and $\int_E (f+g) = \int_E f + \int_E g.$

(Q6) Let g be integrable and $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on E. If $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. on E, then prove that

$$\int_E \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_E f_n \,.$$

(Q7) Consider $f(x) = \dfrac{1}{x}$ on $[0, 8]$. Find truncation function $[f(x)]_n$ for $f(x)$ and use it to calculate the Lebesgue integral of $f(x)$.

(Q8) (a) If f has period 2π and is integrable on $(-\pi, \pi)$, then find $\lim_{n\to\infty} \int f(x) \sin nx \, dx$.

(b) Justify as briefly as possible:

i) Why $\int_0^\infty \frac{\sin x}{x} \, dx$ is not equal to zero.

ii) Why $\lim_{n\to\infty} \int f_n(x) \, dx = 0$ where $f_n(x) = \dfrac{\sin\left(\frac{x^2}{n}\right)}{x}$, $\quad 0 < x < 1$ and $n \geq 1$.

(Q9) Define "Convergence in measure" for a sequence $\{f_n\}$ of measurable functions to a measurable function f on E. Prove that if $f_n \to f$ in measure, then $\alpha f_n \to \alpha_f$ in measure for any real number α.