King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 531 (Real Analysis) **Term 142**

Final Exam: May, 18, 2015 **Time allowed: 3hrs**

Question Number	Marks	Maximum Points
1		3
2		3
3		3
4		3
5		3
6		3
7		3
8		3
9		4
10		4
11		4
12		4
Total		40

1. Prove that (a, ∞) is measurable set for any $a \in \mathbb{R}$. Hence show that any closed set in $(\mathbb{R}, |\cdot|)$ is measurable.

2. Define Lebesgue measure of a set of real numbers. Find Lebesgue measure of:

   i) $A = [-3, 4] \cup (1, 6)$

   ii) $B = \cup_{k=1}^{\infty} \left\{ x \in \mathbb{R} : \frac{1}{2^k} \leq x < \frac{1}{2^{k-1}} \right\}$

3. If $f = g$ *a.e.* on D and f is a Lebesgue measurable function, then prove that g is a measurable function. Hence check whether or not the function

$$g(x) = \begin{cases} 0 & \text{if } x \text{ rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

is measurable.

4. Let $f = \chi_{[-1,1]} + \chi_{[-2,2]} + \chi_{[0,\infty)} - \chi_{(3,\infty)}$.
   Compute:

   (1) Standard representation of this simple function.

   (2) Lebesgue integral of f.

5. Let f be a non-negative measurable function on E. Then show that $\int f = 0$ if and only if $f = 0$ $a.e.$ on E.

6. Let $E = (0,1]$ and $f_n(x) = \begin{cases} n & \text{if } x \in \left(0, \frac{1}{n}\right] \\ 0 & \text{if } x \in \left(\frac{1}{n}, 1\right] \end{cases}.$

   Explain why the conclusion $\left(\int_E \lim_n f_n = \lim_n \int_E f_n\right)$ of Lebesgue dominated convergence theorem fails for the sequence $\{f_n\}$.

7. Let ν be a signed measure on a measurable space (X, β). Prove that there are sets A and B such that
   (i) A is a positive set and B is a negative set.
   (ii) $X = A \cup B$
   (iii) $A \cap B = \phi$.

8. Let μ, ν and λ be $\sigma-$ finite measures. Denote the Radon-Nikodym derivative of ν with respect to μ by $\frac{d\nu}{du}$.
   If $\nu << \mu << \lambda$, then show that

$$\left[\frac{d\nu}{d\lambda}\right] = \left[\frac{d\nu}{d\mu}\right]\left[\frac{d\mu}{d\lambda}\right].$$

9. If f and g are in $L^p\,(p \geq 1)$, then prove that $\parallel f + g \parallel_p \leq \parallel f \parallel_p + \parallel g \parallel_p$.

10. Prove that each function g in L^q defines a bounded linear functional F on L^p by the formula
$$F(f) = \int fg \quad \text{with } \| F \| = \| g \|_q \, .$$

11. a) Show by means of an example that

$$\| f + g \|_{1/2} \geq \| f \|_{1/2} + \| g \|_{1/2} .$$

b) Propose converse of the statement in (Q10) and give name of the basic result needed in its proof. **(Do not give the proof)**.

12. Let f be a bounded and measurable function on $[a, b]$ and

$$F(x) = \int_a^x f(t)dt + F(a).$$

Use bounded convergence theorem to show that $F'(x) = f(x)$ for almost all x in $[a, b]$.