## Math 531 (Real Analysis) Final Exam: May 18, 2015

**Time allowed:** 3hrs  
**Maximum Points:** 40

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1. Prove that $(a, \infty)$ is measurable set for any $a \in \mathbb{R}$. Hence show that any closed set in $(\mathbb{R}, | \cdot |)$ is measurable.
2. Define Lebesgue measure of a set of real numbers. Find Lebesgue measure of:

i) \( A = [-3, 4] \cup (1, 6) \)

ii) \( B = \bigcup_{k=1}^{\infty} \{ x \in \mathbb{R} : \frac{1}{2^k} \leq x < \frac{1}{2^{k+1}} \} \)
3. If \( f = g \) a.e. on \( D \) and \( f \) is a Lebesgue measurable function, then prove that \( g \) is a measurable function. Hence check whether or not the function

\[
g(x) = \begin{cases} 
0 & \text{if } x \text{ rational} \\
1 & \text{if } x \text{ is irrational}
\end{cases}
\]

is measurable.
4. Let \( f = \chi_{[-1,1]} + \chi_{[-2,2]} + \chi_{[0,\infty)} - \chi_{(3,\infty)}. \)
Compute:

(1) Standard representation of this simple function.

(2) Lebesgue integral of \( f. \)
5. Let $f$ be a non-negative measurable function on $E$. Then show that $\int f = 0$ if and only if $f = 0 \ a.e. \ on \ E$. 
6. Let $E = (0, 1]$ and $f_n(x) = \begin{cases} n & \text{if } x \in \left(0, \frac{1}{n}\right] \\ 0 & \text{if } x \in \left(\frac{1}{n}, 1\right] \end{cases}$.

Explain why the conclusion $\left(\int_E f_n = \lim_n \int_E f_n\right)$ of Lebesgue dominated convergence theorem fails for the sequence $\{f_n\}$. 
7. Let \( \nu \) be a signed measure on a measurable space \((X, \beta)\). Prove that there are sets \( A \) and \( B \) such that

(i) \( A \) is a positive set and \( B \) is a negative set.

(ii) \( X = A \cup B \)

(iii) \( A \cap B = \phi \).
8. Let $\mu, \nu$ and $\lambda$ be $\sigma$–finite measures. Denote the Radon-Nikodym derivative of $\nu$ with respect to $\mu$ by $\frac{d\nu}{d\mu}$.
If $\nu << \mu << \lambda$, then show that

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}.$$
9. If $f$ and $g$ are in $L^p (p \geq 1)$, then prove that $\| f + g \|_p \leq \| f \|_p + \| g \|_p$. 
10. Prove that each function $g$ in $L^q$ defines a bounded linear functional $F$ on $L^p$ by the formula

$$F(f) = \int fg \quad \text{with} \quad \|F\| = \|g\|_q .$$
11. a) Show by means of an example that

\[ \| f + g \|_{1/2} \geq \| f \|_{1/2} + \| g \|_{1/2} . \]

b) Propose converse of the statement in (Q10) and give name of the basic result needed in its proof. (Do not give the proof).
12. Let $f$ be a bounded and measurable function on $[a, b]$ and

$$F(x) = \int_a^x f(t)dt + F(a).$$

Use bounded convergence theorem to show that $F'(x) = f(x)$ for almost all $x$ in $[a, b]$. 