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(1) Consider the equation

\[- \Delta u = f, \quad \Omega = (0,1) \times (0,1)\]

with

\[u = 0 \quad \text{on} \quad \Gamma_1\]

and the Robin boundary condition

\[\alpha u + \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_2\]

where \(\partial \Omega = \Gamma_1 \cup \Gamma_2\) and \(\alpha\) is a constant. Derive a variational formulation for this problem and give conditions on \(\alpha\) and \(f\) (if any are required) that guarantee it has a unique solution in \(H^1\). (Hint: multiply by \(v\) and integrate by parts, converting the \(\frac{\partial u}{\partial n} v\) term to \(uv\) using the boundary condition.)
Consider the pure advection equation
\[ u_t + cu_x = 0, \quad -\infty < x < +\infty, t > 0, c > 0 \] (2.1)
subject to
\[ u(x,0) = w(x) \]
Discretized using a 4-point stencil
\[ u^n_j + a_2 u^{n+1}_{j} + a_3 u^{n+1}_{j+1} + a_4 u^{n+1}_{j-1} = 0 \] (2.2)

(a) Find the only \( o(h^2 + k^2) \) accurate scheme for (1.1) in the form (1.2)
(b) Show that this scheme is unconditionally stable.
(3) Consider the equation
\[ u_t + cu_x = 0, \quad -\infty < x < +\infty, t > 0, c > 0 \quad (2.1) \]
with
\[ u(x,0) = w(x) \]
It is easy to show that the solution is given by:
\[ u(x,t) = v(x + ct). \]
(a) State the Courant-Friedrichs-Lewy condition (CFL)
(b) Give an example of a scheme which satisfy CFL condition
(c) Give an example of a scheme which does not satisfy CFL condition
Consider the following initial-boundary value problem for Burger’s equation

Seek \( u = u(x,t) \) defined on \([0,1] \times [0,T]\) such that

\[
\begin{align*}
    u_t + u u_x &= \varepsilon u_{xx} & (x,t) \in (0,1) \times (0,T) \quad \text{where} \quad \varepsilon > 0 \\
    u(x,0) &= v(x) & x \in (0,1) \\
    u(0,t) = u(1,t) &= 0 & t \in (0,T), \quad v(0) = v(1) = 0
\end{align*}
\]

Douglas and B.F. Jones proposed the following predictor-corrector method for the solution of nonlinear parabolic equations, which in our case has the form:

Seek \( \left\{ \hat{U}_j^n \right\}_{j=0,J}, \left\{ U_j^n \right\}_{j=0,J} \) such that

**Predictor:**

\[
\begin{align*}
    1) & \quad \frac{2}{k} (\hat{U}_{j+1}^{n+1} - U_j^n) + U_j^n \frac{U_{j+1}^n - U_{j-1}^n}{2h} = \frac{\varepsilon}{h^2} (\hat{U}_{j+1}^{n+1} - 2\hat{U}_j^{n+1} + \hat{U}_{j-1}^{n+1}), \quad 0 \leq n \leq N-1, \quad 1 \leq j \leq J-1 \\
    \hat{U}_0^{n+1} &= \hat{U}_J^{n+1} = 0, \quad 0 \leq n \leq N-1
\end{align*}
\]

**Corrector:**

\[
\begin{align*}
    \begin{cases}
        U_j^{n+1} - U_j^n \in J \frac{1}{4h} \left[ (U_{j+1}^{n+1} - U_{j+1}^{n+1}) + (U_{j+1}^{n+1} - U_{j-1}^{n+1}) \right] = \\
        2 & \quad \frac{\varepsilon}{2h^2} \left[ (U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) + (U_j^{n+1} - 2U_j^{n+1} + U_j^{n+1}) \right], \quad 0 \leq n \leq N-1, \quad 1 \leq j \leq J-1 \\
        \hat{U}_0^{n+1} = \hat{U}_J^{n+1} = 0, \quad 0 \leq n \leq N-1
    \end{cases}
\end{align*}
\]

With \( U_j^0 = v(x_j), \quad 0 \leq j \leq J \)

So we predict \( \hat{U}_j^{n+1} \) by (1) and then we correct \( U_j^{n+1} \) by (2). Douglas and Jones showed that for \( k \) sufficiently small, \( \max_{j,n} |U_j^n - u(x_j, nk)| = O(h^2 + k^2) \).

(a) Verify that for the solution of the above predictor-corrector scheme, one has to solve two linear tridiagonal systems of equations per time step, one for the predictor and one for the corrector. [Hint: write these two system in matrix form then the two coefficient matrices are tridiagonal]
Consider the equation
\[- \nabla \cdot (a \nabla u) + b \cdot \nabla u + u^2 = f, \quad \Omega = (0,1) \times (0,1)\]
with
\[u = 0 \quad \text{on} \quad \partial \Omega\]
Where the coefficients \( a = a(x,y), b = b(x,y) \) are smooth and such that
\[a(x, y) \geq a_0 > 0, \quad \forall (x, y) \in \Omega\]

(a) Use your knowledge in finite element methods to suggest a weak formulation for this problem.
(b) Suggest a discrete space suitable for this problem [ give a reason for selecting this discrete space ]
(c) After the discretization, Write the resulting algebraic equations in matrix form (note that the system is nonlinear system of equations)
(d) Suggest a method to solve the nonlinear system in (c).
(e) What properties does the Jacobian matrix in (c) has? Symmetric, positive definite, tridiagonal matrix, block-tridiagonal, diagonally dominant, …
(f) Suggest a method to solve a linear system with Jacobian matrix in (e) as a coefficient matrix of the system [ give a reason for selecting your method ]