Problem 1. Let $H = \mathcal{M}^{n \times 1}$ be the space of column vectors ($n$ rows and 1 column) with real entries. We define the following bilinear form

$$< x, y > = y^T Q x,$$

for every $x, y \in \mathcal{M}^{n \times 1}$, $Q \in \mathcal{M}^{n \times n}$ and $y^T$ is the transpose of $y$. Under what conditions on the matrix $Q$ does the above relation define an inner product on $H$?

Problem 2. Define an operator $A$ on $C(0, 1; \mathbb{R})$ by $D(A) = \{ u \in C^2(0, 1; \mathbb{R}); u(0) = u'(0) = 0 \}$ and for $u \in D(A)$, $Au(x) = u''(x) + u(x)$.

Show that $A$ is linear, closed, unbounded and invertible. Find $A^{-1}$.

Problem 3. Consider $A : C(0, 1 : \mathbb{R}) \rightarrow C(0, 1 : \mathbb{R})$ by $Au = \frac{du}{dt}$. Characterize $\sigma(A)$ in the following cases.

(i) $D_1(A) = \{ u \in C(0, 1; \mathbb{R}); u' \in C(0, 1; \mathbb{R}) \}$,

(ii) $D_2(A) = \{ u \in C(0, 1; \mathbb{R}); u' \in C(0, 1; \mathbb{R}), u(0) = 0 \}$,

(iii) $D(A) = \{ u \in C(0, 1; \mathbb{R}); u(0) = u(1) \}$.

Problem 4. Let $H = L^2(-1, 1; \mathbb{R})$. Define $A : H \rightarrow H$ by

$$Au(x) = -\frac{2}{3} u(x) + \int_{-1}^{1} y^2 u(y) dy.$$

Determine $N(A)$, $R(A)$, $A^*$, $N(A^*)$, $R(A^*)$. Show that $R(A)^\perp = N(A^*)$ and $N(A^*)^\perp = R(A)$.

Solve the equation $Au = f$ in the case $f \in R(A)$ and in the case $f \notin R(A)$. 