Q.No.1: (5+5 marks) The distribution of the sample mid-range \( R \) from a uniform distribution of a random variable \( X \) on \( \left( \theta - \frac{1}{2}, \theta + \frac{1}{2} \right) \) has the density function:

\[
f(r) = n 2^{n-1} \left\{ \frac{1}{2} - |r - \theta| \right\}^{n-1}; \quad \theta - \frac{1}{2} < r < \theta + \frac{1}{2}
\]

(a) Show that the sample mid-range is an unbiased estimator of \( \theta \).
(b) Show that the sample mid-range is a consistent estimator of \( \theta \).

Q.No.2: (7 marks) If \( \hat{p}_1 \) is the most efficient estimator of \( p \) and \( \hat{p}_2 \) is a less efficient estimator with relative efficiency \( e = \frac{\text{Var}(\hat{p}_1)}{\text{Var}(\hat{p}_2)} \) and the correlation coefficient between \( \hat{p}_1 \) and \( \hat{p}_2 \) is \( \rho \). If we define another estimator as

\[
\hat{p}_3 = \frac{(1-\rho \sqrt{e})\hat{p}_1 + (e-\rho \sqrt{e})\hat{p}_2}{(1+e-2\rho \sqrt{e})},
\]

then show that \( \rho = \sqrt{e} \).

Q.No.3: (3+3+3 marks) Show that the following densities belong to the exponential family and give the sufficient statistic(s) for the unknown parameter(s).

(a) Negative Binomial distribution

\[
f(x; \theta) = \binom{x + r - 1}{r - 1} p^r (1 - p)^x; \quad x = r, r + 1, r + 2, \ldots
\]

(b) Weibull distribution

\[
f(x; \theta) = \frac{k}{\lambda} \left( \frac{x}{\lambda} \right)^{k-1} e^{-\left( x/\lambda \right)^k}; \quad x > 0
\]

(c) Beta Distribution

\[
f(x; \theta) = \frac{1}{\beta(a,b)} x^{a-1} (1 - x)^{b-1}; \quad 0 < x < 1
\]

where \( \beta(a,b) \) is the beta function defined as \( \beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \).
Q.No.4: (7 marks) In a sequence of \( n \) Bernoulli trials with \( p \) as the probability of success, \( x \) successes were observed. Show that \( \hat{p}\hat{q}^2 \) is a biased estimator of \( pq^2 \) but the bias \( \to 0 \) as \( n \to \infty \), where \( q = 1 - p \), \( \hat{p} = \frac{x}{n} \) and \( \hat{q} = 1 - \hat{p} \).

Q.No.5: (6+4 marks)
(a) Suppose \( X_1, X_2, ..., X_n \) form a random sample from the density function \( f(x; \theta) \), subject to a number of regularity conditions. Find the Cramer-Rao lower bound for the variance of a biased estimator of \( g(\theta) \) where \( g(\theta) \) is some function of the unknown parameter \( \theta \).
(b) Show that
\[
E \left( \frac{\partial}{\partial \theta} \log L(x; \theta) \right)^2 = -E \left( \frac{\partial^2}{\partial \theta^2} \log L(x; \theta) \right)
\]

Q.No.6: (5 marks) If \( X_1, X_2, ..., X_n \) is a random sample from Rayleigh distribution with probability density function \( f(x; \theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}; x > 0 \). Find whether there exist any Minimum Variance Bound (MVB) estimator for \( \theta \). If yes, find its sampling variance as well.

Q.No.7: (8+2 marks) Suppose that 2-dimensional vector \( (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) \) form a random sample from a bivariate normal distribution with probability density function:
\[
f(x; \theta) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)} \left( \left( \frac{x-\mu_1}{\sigma_1} \right)^2 + \left( \frac{y-\mu_2}{\sigma_2} \right)^2 - 2\rho \left( \frac{x-\mu_1}{\sigma_1} \right) \left( \frac{y-\mu_2}{\sigma_2} \right) \right)}; \ -\infty < x < \infty \quad \text{and} \quad -\infty < y < \infty
\]
where \( \mu_1 \) and \( \mu_2 \) are the unknown means of \( X \) and \( Y \) respectively, \( \sigma_1^2 \) and \( \sigma_2^2 \) are the known variances of \( X \) and \( Y \) respectively and the correlation coefficient (\( \rho \)) between \( X \) and \( Y \) is also known.
(a) Find the Maximum Likelihood Estimators (MLE) of \( \mu_1 \) and \( \mu_2 \).
(b) Comment on the 4 basic properties (unbiasedness, consistency, sufficiency, efficiency) of the estimators found in part (a).

Q.No.8: (7 marks) Suppose that \( Y \) follows exponential distribution with probability density function
\[
f(y; \theta) = \theta e^{-\theta y}; y > 0. \text{ We observe } y_1 = 3, y_2 = 6, y_3 = 10 \text{ and the prior of } \theta \text{ is } h(\theta) = \frac{3\theta^2}{19} ; 2 < \theta < 3. \text{ Find the expression for Bayes’ estimate of } \theta.
\]

With the Best Wishes