Instructions:

1. Write neatly and legibly. You may lose points for messy work.

2. Show all your work. No points for answers without justification.

3. Calculators and Mobiles are not allowed.

4. Make sure that you have 8 different problems

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Q1) Classify the given partial differential equation as hyperbolic, parabolic, or elliptic

(a) $\frac{\partial^2 u}{\partial x^2} - 3 \frac{\partial^2 u}{\partial x \partial y} - 2 \frac{\partial^2 u}{\partial y^2} = 0$

(b) $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

Q2) Solve the heat equation

subject to the given conditions

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \quad t > 0$

$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0$

$u(x,0) = 100, \quad 0 < x < \pi$
Q3 ) Find the Fourier series expansion of the function \( f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x^2 & 0 \leq x < \pi \end{cases} \)

What does the series converge to when \( x = 0 \)?
Q4) Expand \( f(x) = x, \ -\pi < x < \pi \) in a sine series.
Q5) Write out the first three nonzero terms in the Fourier-Legendre expansion of the function

\[ f(x) = \begin{cases} 
0, & -1 < x < 0 \\
2x, & 0 \leq x < 1 
\end{cases} \]
Q6) Consider the Parametric Bessel Equation:

\[ x^2 y'' + x y' + (\alpha^2 x^2 - 4) y = 0 \]

Subject to: \( y \) is bounded on \([0,5]\), \( y(5) = 0 \)

a) State the eigenvalues and the eigenfunctions.

b) Put the differential equation in self-adjoint form

c) Give the orthogonality relation.
Q7  a) show that the set of functions
\[ \{ \sin nx \}, n = 1, 2, 3, \ldots \]
is orthogonal on the interval \([0, \pi]\).

b) Find the norm of each function in the set.

b) Use the orthogonal set given in part (a) to construct an orthonormal set.
Q8) Expand \( f(x) = x^2, \ 0 < x < 1, \) in a Fourier-Bessel series, using Bessel functions of order two that satisfy the boundary condition \( J_2(\alpha) = 0. \)