7:00–09:00 pm

Name : ........................................... ........... ...........

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Exercise 1 (10 pts). Let $n \in \mathbb{Z}$ such that $n \not\equiv 2 \pmod{5}$ and $n \not\equiv 3 \pmod{5}$. Show that

$$5 \mid (4n^2 + 1) \iff 5 \nmid n.$$
Exercise 2 (10 pts). Let $a, b \in \mathbb{Z}$. Show that if $a \equiv 4 \pmod{5}$ and $b \equiv 3 \pmod{4}$, then $(4a + 5b) \equiv 1 \pmod{10}$. 
Exercise 3 (10 pts). Show that if $r$ is a real number such that $0 < r < \frac{2}{3}$, then
\[
\frac{1}{r(2-3r)} \geq 3.
\]
Exercise 4 (8 pts). Write the addition and multiplication tables for $\mathbb{Z}_{10}$.
Solve the equation $3x \equiv 1 \pmod{10}$. 
Exercise 5 (12 pts).

(a) Show that $\sqrt{10}$ is irrational.

(b) Show that if

\[ S = \{ p + q\sqrt{5} \mid p, q \in \mathbb{Q} \} \quad \text{and} \quad T = \{ r + s\sqrt{2} \mid r, s \in \mathbb{Q} \}, \]

then $S \cap T = \mathbb{Q}$.
Exercise 6 (8 pts). Show that there exist no positive integers $m, n$ such that

$$m^2 + m + 1 = n^2.$$
Exercise 7 (12 pts). Let $\mathbb{N} = \{0, 1, 2, 3, \ldots \}$ be the set of all natural numbers. We define the relation $R$ on the cartesian product $\mathbb{N} \times \mathbb{N}$ by:

$$(x, y)R(a, b) \iff x + b = y + a.$$  

(i) Show that $R$ is an equivalence relation.

(ii) If $x, y \in \mathbb{N}$, then we denote by $[x, y]$ the equivalence class of $(x, y)$ with respect to the equivalence relation $R$. Define the addition $\oplus$ on the quotient set $\mathcal{Z} := (\mathbb{N} \times \mathbb{N})/R$ by:

$$[x, y] \oplus [a, b] = [x + a, y + b],$$

and the multiplication $\otimes$ by:

$$[x, y] \otimes [a, b] = [xa + yb, xb + ya].$$

For $x \in \mathbb{N}$, we denote by $-x = (0, x)$ and $x = (x, 0)$.

• Explain why $-2 \oplus -3 = -5$?

• Explain why $-2 \otimes -3 = 6$?

(iii) Show that

$$\mathcal{Z} = \left\{ -x : x \in \mathbb{N} \right\} \cup \left\{ x : x \in \mathbb{N} \right\}.$$
Exercise 8 (10 pts). Using Mathematical induction, show that $3^{2n} - 1$ is a multiple of 8, for all integers $n \geq 0$. 