King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math-280, Term-151
Major Exam 2, Time Allowed: 2 hours

Name: ID:

SHOW ALL YOUR WORK

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Question 1: Let $V$ be a vector space. Show that
   a) the element $0$ in $V$ is unique.
   b) $\alpha 0 = 0$ for each scalar $\alpha$.  
Question 2: Let $S_1$ and $S_2$ be two subspaces of a vector space $V$. Determine whether the following sets are subspaces of $V$.

a) $S_1 \cap S_2$

b) $S_1 \setminus S_2 = \{ v \in S_1 : v \notin S_2 \}$. 
Question 3: Check for linear independence

a) \[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}
\] in \( \mathbb{R}^{2 \times 2} \).

b) \( x_2 - x_1, \quad x_3 - x_2, \quad x_3 - x_1 \) in \( \mathbb{R}^n \) where \( x_1, x_2, x_3 \) are linearly independent vectors in \( \mathbb{R}^n \).
Question 4:

a) Find a basis for the subspace $S$ of $\mathbb{R}^4$ that consists of all vectors of the form $(a-2b,a-b-3c,b,a)^T$, where $a,b,c \in \mathbb{R}$. What is the dimension of $S$?

b) In $C[-\pi,\pi]$, what is the dimension of $\text{Span}(1,\cos x,\sin^2(\frac{x}{2}))$.
Question 5: Let $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $u_3 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

a) Show that the set $B_1 = \{u_1, u_2, u_3\}$ forms a basis for $\mathbb{R}^3$.

b) Find the transition matrix from the $B_2 = \{e_1, e_2, e_3\}$ to $B_1$.

c) Using part b) find $[v]_{B_1}$ where $v = (3, 2, -5)^T$. 
Question 6: For the matrix

\[ A = \begin{pmatrix}
3 & 1 & -3 & 4 \\
-1 & 2 & 1 & -2 \\
-3 & 8 & -4 & 2 \\
\end{pmatrix} \]

find

a) a basis for
i. the row space of \( A \),
ii. the column space of \( A \), and
iii. the null space of \( A \).

b) \( \text{rank}(A) \), \( \text{nullity}(A) \).