King Fahd University of Petroleum and Minerals  
Department of Mathematics and Statistics

Math 301 Final Exam  
Semester (151)  
Dec. 27, 2015 at 07:00-10:00 PM

Name: ...........................................................................................................

I.D: ...................................... Section: ....... Serial: ........

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<th>Question</th>
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Consider the following Sturm-Liouville problem.

\[ x^2 y'' + 5xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0. \]

a. Put the equation in the self-adjoint form.

b. Find out the weight function.

c. Write the orthogonality relation.
The steady-state temperature in a square plate modeled by the following BVP. Use the method of separation of variables to find $u(x, y)$.

$$\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \quad 0 < x < \pi, \quad 0 < y < \pi \\
u_x(0, y) &= 0, \quad u_x(\pi, y) = 0 \\
u(x, 0) &= 0, \quad u(x, \pi) = x
\end{align*}$$
Question 3

The displacement $u(r, t)$ of a circular membrane of radius $c = 3$ clamped along its circumference with initial displacement and initial velocity in polar coordinates modeled by

\[
\begin{align*}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} &= \frac{\partial^2 u}{\partial t^2}, & 0 < r < 3, & t > 0, \\
u(3, t) &= 0, & t > 0, \\
u(r, 0) &= 1, & \frac{\partial u}{\partial t}\bigg|_{t=0} &= \frac{1}{2}.
\end{align*}
\]

Using separation of variables to find $u(r, t)$. Note that $u$ is bounded at $r = 0$. 
CONT...Q3
Question 4

Using **Cylindrical** Coordinates, the steady-state temperature $u(r, z)$ in a circular cylinder subject to conditions given by the following system.

$$
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < 2, \quad 0 < z < 4,
$$

$$
u(2, z) = 0, \quad 0 < z < 4,
$$

$$
u(r, 0) = 0, \quad u(r, 4) = c_0 \quad 0 < r < 2.
$$

Find $u(r, z)$. Note that $u$ is bounded at $r = 0$. 
Using the Laplace Transform to find the displacement $u(x, t)$ of a very long string in the following model.

$$\frac{\partial^2 u}{\partial x^2} - 10 = \frac{\partial^2 u}{\partial t^2}, \quad x > 0, \quad t > 0,$$

subject to

$$\begin{cases} 
    u(0, t) = 0, & \lim_{x \to \infty} \frac{\partial u}{\partial x} = 0, \quad t > 0, \\
    u(x, 0) = 0, & \frac{\partial u}{\partial t} \bigg|_{t=0} = 0, \quad x > 0.
\end{cases}$$
CONT...Q5
Question 6 (10+5 points)

a. Represent \( f(x) = e^{-2x}, \ x > 0 \) by Fourier cosine integral.

b. Use the result in (a) to evaluate \( \int_{0}^{\infty} \frac{\cos t}{4+t^2} \, dt \).
Question 7 (22 points)

Use the **Fourier sine** to find the temperature $u(x, t)$ in a semi-infinite rod modeled by

$$
\begin{align*}
\left\{ \begin{array}{ll}
k \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}, & x > 0, \ t > 0, \\
 \frac{\partial u}{\partial x} (0, t) &= 0, & t > 0, \\
 u(x, 0) &= f(x) = \begin{cases} 
 1, & 0 < x < 1 \\
 0, & x > 1.
\end{cases}
\end{array} \right.
\end{align*}
$$