Q.1: (25 points) Use Laplace transform to solve $u_{xx} = u_{tt} - xe^{-t}$, $0 < x < \infty$, $t > 0$

under the following conditions

$u(0, t) = \cos t$, and $\lim_{x \to \infty} |u(x, t)| \sim x^n$ for some $n$ and $t > 0$

$u(x, 0) = 1$ and $u_t(x, 0) = 0$, for $0 < x < \infty$

Q.2: (25 points) Use Hankel transform to solve the wave equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{1}{C^2} \frac{\partial^2 u}{\partial t^2}$$

$0 < r < \infty$, $t > 0$,

subject to the initial conditions

$u(r, 0) = f(r)$

$u_t(r, 0) = g(r)$.

Q.3: (25 points) Show that $\mathcal{H}\{f'(r)\} = \frac{\alpha}{2n} \left[ (n-1)\tilde{f}_{n+1}(\alpha) - (n+1)\tilde{f}_{n-1}(\alpha) \right]$, $n \geq 1$

Q.4: (25 points) Solve using Mellin transform

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \leq x < \infty, \quad 0 < y < 1,$$

subject to the conditions

$u(x, 0) = 0$

$u(x, 1) = \begin{cases} A, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$

Q.5: (20 points) Find and sketch image of the vertical strip $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ under the mapping $w = f(z) = \sin z$. Check if $f$ is conformal or not.

Q.6: (20 points) Find a harmonic function $\Phi(x, y)$ in the upper half of the $z$-plane which satisfy

$\Phi(x, 0) = G(x) = \begin{cases} A, & x > 0 \\ 0, & x < 0 \end{cases}$