Exercise 1

Prove that

\[ \tan^{-1} z = \frac{1}{2i} \log\left( \frac{1 + zi}{1 - zi} \right). \]

Deduce all the possible values of \( \tan^{-1}(1) \).
Exercise 2
Choose the constant $a$ so that the function $u(x, y) = ax^2y - y^3 + xy$ is harmonic, and find all its harmonic conjugates.
Exercise 3
Evaluate the following two integrals

\[ \oint_{|z|=5} \frac{e^z}{(z-2)(z-4)} \, dz \quad \text{and} \quad \oint_{|z|=1} z^k \cos \frac{1}{z} \, dz \quad (k \in \mathbb{Z}) \]
Exercise 4

(i) State Cauchy’s estimate for derivatives.

(ii) By considering \( f(z) = e^z \) and the circle \( |z| = n \), prove the inequality

\[
\left( \frac{n}{e} \right)^n \leq n! \text{ for } n \in \mathbb{N}, n \geq 1.
\]
Exercise 5
Let $f$ and $g$ be two analytic functions in a domain $\Omega$ such that $f(z)g(z) = 0$ in $\Omega$. Prove that either $f(z) = 0$ or $g(z) = 0$ in $\Omega$. 
Exercise 6

1. Show that \( \int_{|w|=r} \frac{dw}{w^n(w-z)^2} = 0 \), for \( |z| < r \) and \( n \geq 0 \). (Hint: use the substitution \( u = 1/w \))

2. Deduce that \( \int_{|w|=r} \frac{f(w)}{(w-z)^2} dw = 0 \), for \( |z| < r \) and any analytic function \( f \) on \( |w| \leq r \).

3. Find \( \int_{|w|=r} \frac{\text{Re} f(w)}{(w-z)^2} dw \), for \( |z| < r \).

4. Find \( \int_{|w|=r} \frac{\text{Re} f(w)}{w-z} dw \), for \( |z| < r \).