King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH533 - Complex Variables
Midterm Exam – Semester I, 2015-2016

Exercise 1

Prove that

$$\tan^{-1} z = \frac{1}{2i} \log(\frac{1+zi}{1-zi}).$$

Deduce all the possible values of $\tan^{-1}(1)$.

Exercise 2

Choose the constant a so that the function $u(x, y) = ax^2y - y^3 + xy$ is harmonic, and find all its harmonic conjugates.

Exercise 3

Evaluate the following two integrals

$$\oint_{|z|=5} \frac{e^z}{(z-2)(z-4)}\,dz \quad \text{and} \qquad \oint_{|z|=1} z^k \cos\frac{1}{z}\,dz \quad (k \in \mathbb{Z})$$

Exercise 4

(i) State Cauchy's estimate for derivatives.

(ii) By considering $f(z) = \mathrm{e}^z$ and the circle $|z| = n$, prove the inequality

$$\left(\frac{n}{e}\right)^n \leq n! \text{ for } n \in \mathbb{N}, n \geq 1.$$

Exercise 5

Let f and g be two analytic functions in a domain Ω such that $f(z).g(z) = 0$ in Ω. Prove that either $f(z) = 0$ or $g(z) = 0$ in Ω.

Exercise 6

1. Show that $\displaystyle\int_{|w|=r} \frac{dw}{w^n(w-z)^2} = 0$, for $|z| < r$ and $n \geq 0$. (Hint: use the substitution $u = 1/w$)
2. Deduce that $\displaystyle\int_{|w|=r} \frac{\overline{f}(w)}{(w-z)^2}\, dw = 0$, for $|z| < r$ and any analytic function f on $|w| \leq r$.
3. Find $\displaystyle\int_{|w|=r} \frac{\text{Re } f(w)}{(w-z)^2}\, dw$, for $|z| < r$.
4. Find $\displaystyle\int_{|w|=r} \frac{\text{Re } f(w)}{w-z}\, dw$, for $|z| < r$.