King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
(Math 260)

First Major Exam
Term 152
Monday, February 22, 2016
Net Time Allowed: 100 minutes

Name:	
ID:	
Section No:	
Instructor's Name	

(Show all your steps and work)

Question #	Marks
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(1) Solve the first order initial value problem $\dfrac{dy}{dx} = xe^{-2x}$; $y(0) = 2$.

Also, give the interval over which the solution of the initial value problem is defined.

[10 points]

We integrate the differential equation

$$\int \frac{dy}{dx}\,dx = \int x e^{-2x}\,dx$$

$$\Rightarrow \quad y(x) = \int x e^{-2x}\,dx$$

Using integration by parts with $u = x$, $du = dx$, $v = -\frac{1}{2}e^{-2x}$, $dv = e^{-2x}dx$

$$y(x) = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$$

$$y(x) = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c \quad \text{or} \quad y(x) = -\frac{1}{4}(2x+1)e^{-2x} + c$$

We use the initial condition $y(0) = 2$ to find c.

$$2 = -\frac{1}{4}(2.0+1)e^{0} + c \quad \Rightarrow c = \frac{9}{4}$$

Then the solution of the initial value problem is

$$y(x) = -\frac{1}{4}(2x+1)e^{-2x} + \frac{9}{4}$$

The interval over which the solution is defined is $(-\infty, \infty)$.

(2) An object is to be cooled by setting it outside where the temperature is $15\,^{o}c$. Suppose that the temperature of the object has dropped to $42\,^{o}c$ after 1 minute and to $23\,^{o}c$ after 4 minutes. What was the initial temperature of the object? [12 points]

The temperature function is $T(t) = (T_0 - A)e^{kt} + A$ where A is the temperature of the surrounding and T_0 is the initial temperature.

In our problem $A = 15\,^{\circ}C$ and we want to find T_0.

We are given the following information $T(1) = 42\,^{\circ}C$ and $T(4) = 23\,^{\circ}C$.

$$T(1) = (T_0 - 15)e^{k} + 15 = 42 \Rightarrow e^{k} = \frac{27}{T_0 - 15}$$

$$T(4) = (T_0 - 15)e^{4k} + 15 = 23 \Rightarrow e^{4k} = \frac{8}{T_0 - 15}$$

$$\Rightarrow \left(\frac{27}{T_0 - 15}\right)^4 = \frac{8}{T_0 - 15} \Rightarrow (T_0 - 15)^3 = \frac{27^4}{8}$$

$$\Rightarrow T_0 - 15 = \frac{3^4}{2} \Rightarrow T_0 = \frac{3^4}{2} + 15 = 55.5\,^{\circ}C.$$

(3) Solve the initial value problem $y\frac{dy}{dx} = \tan x \ \csc y,\ y(\pi) = \frac{\pi}{2}$. [10 points]

We can solve this problem using separation of variables.

$$y \sin y \, dy = \tan x \, dx.$$

Upon integrating both sides

$$\int y \sin y \, dy = \int \tan x \, dx \Rightarrow \int y \sin y \, dy = \ln|\sec x| + C$$

To calculate $\int y \sin y \, dy$, we use integration by parts with

$u = y \qquad v = -\cos y$

$du = dy \qquad dv = \sin y \, dy.$

Then $\int y \sin y \, dy = -y\cos y + \int \cos y \, dy = -y\cos y + \sin y + C.$

The general solution of this differential equation is

$$-y\cos y + \sin y = \ln|\sec x| + C.$$

We use the initial condition $y(\pi) = \pi/2$ to find C

$$\underbrace{-\frac{\pi}{2}\cos\frac{\pi}{2}}_{=0} + \underbrace{\sin\frac{\pi}{2}}_{1} = \underbrace{\ln|\underbrace{\sec \pi}_{=-1}|}_{0} + C \Rightarrow C = 1$$

Finaly, the solution of the initial value problem is

$$-y\cos y + \sin y = \ln|\sec x| + 1$$

(4) Solve the first order initial value problem $(x-1)\dfrac{dy}{dx}+2xy=\dfrac{xe^{-2x}}{x-1}$; $y(2)=e^{-4}$.

Also, give the interval over which the solution of the IVP is defined. [12 points]

We rewrite the differential equation as

$$\frac{dy}{dx}+\frac{2x}{x-1}y=\frac{xe^{-2x}}{(x-1)^2}$$

This is a 1^{st} order linear differential equation with

$$P(x)=\frac{2x}{x-1} \qquad x\neq 1.$$

We find the integrating factor:

$$\rho(x)=e^{\int\frac{2x}{x-1}dx}=e^{\int 2dx+2\int\frac{dx}{x-1}}=e^{2x+2\ln|x-1|}=(x-1)^2e^{2x}.$$

We use the integrating factor to rewrite the diff. eqn.

$$\left[(x-1)^2e^{2x}y\right]'=(x-1)^2e^{2x}\,\frac{xe^{-2x}}{(x-1)^2}$$

$$\Rightarrow \left[(x-1)^2e^{2x}y\right]'=x$$

We integrate both sides:

$$(x-1)^2e^{2x}y=\frac{x^2}{2}+C.$$

We use the initial condition $y(2)=e^{-4}$ to find C.

$$(2-1)^2e^4e^{-4}=\frac{4}{2}+C \Rightarrow C=-1.$$

Then the solution of the I.V.P is $(x-1)^2e^{2x}y=\frac{x^2}{2}-1$

explicitely: $y(x)=\dfrac{(x^2-2)}{2\cdot(x-1)^2e^{2x}}$

The interval over which the solution is defined is $(-\infty,1)\cup(1,\infty)$

(5) Consider the differential equation [10 points]

$$\frac{dy}{dx}=\frac{3y-2x+4}{4x+2y-8}.$$

Find h and k so that the substitutions $x=u+h$ and $y=v+k$ transform the above differential equation into a homogenous differential equation of the form $\frac{dv}{du}=F\left(\frac{v}{u}\right)$.

$3y-2x+4=3v+3k-2u-2h \Rightarrow 3k-2h=4$

$4x+2y-8=4u+4h+2v+2k \Rightarrow 4h+2k=-8$

The solution of the above linear system is $h=-2$ and $k=0$.

Then $\dfrac{3y-2x+4}{4x+2y-8}=\dfrac{3v-2u}{4u+2v}$

We rewrite $\frac{dy}{dx}$ as follows: $\frac{dy}{dx}=\frac{dv}{du}\frac{du}{dx}=\frac{dv}{du}\frac{d}{dx}(x-2)$

Then $\dfrac{dy}{dx}=\dfrac{dv}{du}$

The differential equation can be written in terms of u and v as:

$$\frac{dv}{du}=\frac{3v-2u}{4u+2v}.$$

which can be rewritten as

$$\frac{dv}{du}=\frac{3\left(\frac{v}{u}\right)-2}{4+2\left(\frac{v}{u}\right)}.$$

(6) Solve the differential equation [12 points]

$$2xy^2dx + 2x^2ydy = -3x^2dx - 4y^3dy.$$

We rewrite the differential equation as

$$(2xy^2 + 3x^2)\,dx + (2x^2y + 4y^3)\,dy = 0$$

Let $M(x,y) = 2xy^2 + 3x^2$ and $N(x,y) = 2x^2y + 4y^3$.

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 4xy. \\ \frac{\partial N}{\partial x} &= 4xy \end{aligned}\right\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{the differential eqn is exact.}$$

Let $F(x,y) = c$ be solution of this differential equation.

Then $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$.

$$F(x,y) = \int 2xy^2 + 3x^2\,dx = x^2y^2 + x^3 + g(y)$$

$$\frac{\partial F}{\partial y} = N \Rightarrow 2x^2y + g'(y) = 2x^2y + 4y^3 \Rightarrow g'(y) = 4y^3$$

$$g(y) = \int 4y^3\,dy = y^4 + c'$$

Then $F(x,y) = x^2y^2 + x^3 + y^4 + c'$ and the general solution is $x^2y^2 + x^3 + y^4 = c''$.

(7) The general solution of the second order differential equation [10 points]

$$y'' - 4y' + 4y = 0 \text{ is } y(x) = (c_1 + c_2 x)e^{2x}.$$

Find the values of c_1 and c_2 so that $y(x)$ satisfies the initial conditions $y(0) = 2$ and $y'(0) = 3$.

Consider $y(x) = (c_1 + c_2 x)e^{2x}$.

Then $y'(x) = (2c_2 x + 2c_1 + c_2)e^{2x}$

Using the initial conditions $y(0) = 2$ and $y'(0) = 3$, we find

$$\begin{cases} c_1 = 2 \\ 2c_1 + c_2 = 3 \end{cases}$$

Then $c_1 = 2$ and $c_2 = -1$.

(8) Determine the value (s) of k for which the system [10 points]

$$4x+3y=5$$
$$8x+ky=10$$

has:

a) No solution,
b) Unique solution,
c) Infinitely many solutions.

The augmented coefficient matrix of this system is

$A = \begin{bmatrix} 4 & 3 & 5 \\ 8 & k & 10 \end{bmatrix}$. We rewrite it in echelon form.

$$\begin{bmatrix} 4 & 3 & 5 \\ 8 & k & 10 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 4 & 3 & 5 \\ 0 & k-6 & 0 \end{bmatrix}$$

Case a): This system will never have "No Solution".

Case b): For all $k \neq 6$, this system will have "Unique Solution

Case c): For $k=6$, this system will have "Infinitely Many Solution

(9) Using Gauss-Jordan elimination method, solve the following linear system: [14 points]

$$x_1 - x_2 + 3x_4 + x_5 = 2$$
$$x_1 + x_2 + 2x_3 + x_4 - x_5 = 4$$
$$x_2 + 2x_4 + 3x_5 = 0$$

The augmented coefficient matrix of the system is

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{bmatrix}.$$

Using Gauss-Jordan elimination method means, we shall find the reduced echelon form of A.

$$\begin{bmatrix} 1 & -1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 2 & 2 & -2 & -2 & 2 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{R_2 = R_2/2}$$

$$\begin{bmatrix} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 3 & 4 & -1 \end{bmatrix} \xrightarrow{R_3 = -R_3} \begin{bmatrix} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & -3 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 = R_2 - R_3} \begin{bmatrix} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & -3 & -4 & 1 \end{bmatrix} \xrightarrow{R_1 = R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 & 5 & 4 & 2 \\ 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & -3 & -4 & 1 \end{bmatrix}.$$

The linear system that corresponds to this reduced echelon form is

$$\begin{cases} x_1 + 5x_4 + 4x_5 = 2 \\ x_2 + 2x_4 + 3x_5 = 0 \\ x_3 - 3x_4 - 4x_5 = 1 \end{cases}$$

Then setting $x_4 = s$ and $x_5 = t$, we have $x_3 = 1 + 3s + 4t$

$x_2 = -2s - 3t$

$x_1 = 2 - 5s - 4t.$

Question 9 (cont'd)

The solution set of this linear system is

$$\{(x_1, x_2, x_3, x_4, x_5) \mid x_1 = 2-5s-4t, x_2 = -2s-3t, x_3 = 1+3s+4t, x_4 = s, x_5 = t\}$$