1. Write clearly.

2. Show all your steps.

3. No credit will be given to wrong steps.

4. Do not do messy work.

5. Calculators and mobile phones are NOT allowed in this exam.

6. Turn off your mobile.
1. Let \( A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & -2 \\ 2 & 2 & 4 \end{bmatrix} \)

(a) Find elementary matrices \( E_1, E_2, E_3 \) and \( P \) such that
\[
E_3 E_2 E_1 P A = U.
\]

(b) Using (a) compute the LU factorization of the matrix \( PA \) (i.e. show that \( A = PLU \))

(c) Using (b), solve the system \( Ax = b \), where \( b = (-3, 1, 0) \).
2. Find bases and their dimensions for the four fundamental subspaces of

\[ A = \begin{bmatrix}
1 & 2 & 1 & 3 & 3 \\
2 & 4 & 0 & 4 & 4 \\
1 & 2 & 3 & 5 & 5 \\
2 & 4 & 0 & 4 & 2
\end{bmatrix} \]
3. Let \( \mathcal{X} \) and \( \mathcal{Y} \) are two subspaces of a vector space \( \mathcal{V} \), show the following

(a) \( \mathcal{X} + \mathcal{Y} = \{ x + y \mid x \in \mathcal{X}, y \in \mathcal{Y} \} \) is a subspace of \( \mathcal{V} \).

(b) \( \dim (\mathcal{X} + \mathcal{Y}) = \dim(\mathcal{X}) + \dim(\mathcal{Y}) - \dim(\mathcal{X} \cap \mathcal{Y}) \)

(c) For any conformal matrices \( A \) and \( B \) (use part(b)) to show

\[
\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)
\]
4. Identify all subspaces of $\mathbb{R}^2$ that are invariant under $A = \begin{bmatrix} -9 & 4 \\ -24 & 11 \end{bmatrix}$
5. If \( P \) is the projector that maps each \( v = (x, y, z) \in \mathbb{R}^3 \) to its orthogonal projection \( P(v) = (x, y, 0) \) in the \( xy \)-plane.

(a) Calculate \([P]_B\) where the bases \( B \) and \( B' \) are

\[
B = \left\{ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}
\]

and

\[
B' = \left\{ v_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}
\]

(b) Find \([P(w)]_{B'}\), where \( w = (-1, 2, -1) \).
6. Find $A^{3000}$, given

$$A = \begin{bmatrix}
1 & 0 & 0 & 1/3 & 1/3 & 1/3 \\
0 & 1 & 0 & 1/3 & 1/3 & 1/3 \\
0 & 0 & 1 & 1/3 & 1/3 & 1/3 \\
0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\
0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\
0 & 0 & 0 & 1/3 & 1/3 & 1/3
\end{bmatrix}$$

Hint: A square matrix $C$ is said to be idempotent when it has the property that $C^2 = C$. Make use of idempotent submatrices in $A$. 
7. Let $S$ be a skew-symmetric matrix with real entries.

(a) Prove that $(I - S)$ is nonsingular. **Hint:** $x^Tx = 0 \rightarrow x = 0$.

(b) If $A = (I + S)(I - S)^{-1}$, show that $A^{-1} = A^T$. 