King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 531 (Real Analysis) - Final Exam - Term 152

Duration: 180 minutes

Name:______________________________ ID Number:____________________________

Section Number:____________________ Serial Number:__________________________

Class Time: ________________________ Instructor's Name:______________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 12 pages of problems (Total of 12 Problems)

Question Number	Points	Maximum Points
1		7
2		15
3		10
4		18
5		10
6		10
7		20
8		10
Total		100

1. Define a Lebesgue measurable set. Show that the class $\mathfrak{M}$ of Lebesgue measurable sets is a $\sigma-$ algebra.

2. (i) Define Cantor ternary set C.

(ii) Find Lebsgue measure of C.

(iii) Show that C in uncountable.

3. Let $\{f_n\}$ be a sequence of extended real-valued measurable functions with the same domain D. Prove that $\underline{\lim}\ f_n$ and $\{x \in D : f_1(x) < f_2(x)\}$ are measurable.

4. The measurable function of a measurable function is not measurable.
Justify this statement.

5. Let (X, β) be a measurable space and f be a real-valued function on X. If $f^{-1}(O)$ is measurable for each open set O of real numbers, then show that f is measurable.

6. Let $E = (-\infty, \infty)$, $f_n(x) = e^{-nx^2+x}$ and

$$g(x) = \begin{cases} e & 0 \leq x \leq 2 \\ e^{-|x|} & \text{otherwise} \end{cases}$$

Then show:

(i) $f_n(x) \leq g(x)$ for all n

(ii) g and f_n are integrable functions.

(iii) by means of Lebesgue dominated convergence theorem that $\lim_n \int_E f_n = \int_E \lim_n f_n$.

7. State and prove Jordan decomposition theorem.

8. Let (X, β, μ) be a σ- finite measure space and ν be a measure on (X, β) such that $\nu << \mu$. Then prove that there is a non-negative measurable function f on X such that

$$\nu(E) = \int_E f \, d\mu \quad \text{for all}\, E \in \beta.$$

Justify that the function f is unique in $[\mu]$ a.e. sense.

9. Use the information given in (Q8) to show that:

(a) $\int f \, d\nu = \int f \left[\frac{d\nu}{d\mu}\right] \, d\mu$

(b) $\left[\frac{d(\nu_1 + \nu_2)}{d\mu}\right] = \left[\frac{d\nu_1}{d\mu}\right] + \left[\frac{d\nu_2}{d\,\mu}\right]$

10. If $f \in L^p$ and $g \in L^q$ (where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$), then show that

$$||f\,g||_1 = \int |f\,g|\,d\mu \leq ||f||_p\,||g||_q.$$

11. (a) Let μ and ν be finite measures on a measurable space (X, β). If $\lambda = \mu + \nu$ and $F(f) = \int f \, d\mu$, then show that F is a well-defined and bounded linear functional on $L^2(\lambda)$.

(b) Let g be an integrable function and M be a constant such that $|\int f\, g| \leq M$ for all bounded measurable functions f.
Then show that g is in L^q and $||g||_q \leq M$ where q is as in $(Q10)$.

12. Prove that $L^p (1 < p < \infty)$ is a Banach Space under the norm

$$||f||_p = \left(\int |f|^p \, d\mu \right)^{1/p}.$$