(1) [10 points] Let \( \{ R_i \}_{i \in \Delta} \) be a family of rings and consider the ring \( R := \prod_{i \in \Delta} R_i \). Prove that if \( \Delta \) is finite, then every ideal of \( R \) has the form \( \bigoplus_{i \in \Delta} I_i \) for some ideal \( I_i \) of \( R_i \) (\( i \in \Delta \)). Is this fact true if \( \Delta \) is infinite? (Justify)

(2) [10 points] Let \( R \) be a ring. Prove:
\[
0 \to Y' \to Y \to Y'' \text{ exact} \iff 0 \to \text{Hom}_R(X, Y') \to \text{Hom}_R(X, Y) \to \text{Hom}_R(X, Y'') \text{ exact}, \ \forall X
\]

(3) [10 points] Let \( R \) be a commutative ring and let \( M \) be an \( R \)-module. Prove the implications:
\[
M \text{ free} \Rightarrow M \text{ projective} \Rightarrow M \text{ flat}
\]
and provide explicit counterexamples for the converses.

(4) [10 points] Prove that if a short exact sequence of modules \( 0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0 \) splits, then:
\begin{enumerate}
  
  (a) [4 points] \( M = \text{Ker}(g) \oplus \text{Im}(f) \), where \( g \circ f = 1_{M''} \).

  (b) [4 points] \( M = \text{Ker}(\psi) \oplus \text{Im}(f) \), where \( \psi \circ f = 1_{M''} \).

  (c) [2 points] \( M \cong M' \oplus M'' \).
\end{enumerate}

(5) [15 points] Let \( R \) be an integral domain containing a field \( k \). Prove that if \( R \) is a finite dimensional vector space over \( k \), then \( R \) is a field.

(6) [20 points] Let \( R \) be a commutative ring. An \( R \)-module \( M \) is called faithfully flat if \( M \) is flat and \( M \otimes N = 0 \Rightarrow N = 0 \). Prove that the following conditions are equivalent:
\begin{enumerate}
  
  (i) \( M \) is faithfully flat;

  (j) \( M \) is flat and if \( u: E \to F \) is an \( R \)-map with \( u \neq 0 \), then \( T_M(u): M \otimes E \to M \otimes F \) is also \( \neq 0 \);

  (k) \( M \) is flat and \( pM \neq M \) for all maximal ideals \( p \) of \( R \);

  (l) \( N' \to N \to N'' \) is exact \( \iff M \otimes N' \to M \otimes N \to M \otimes N'' \) is exact.
\end{enumerate}
[Each implication = 5 points]

(7) [25 points] Let \( R \) be a commutative ring and \( E, F \) two \( R \)-modules. We say that \( E \) is \( F \)-flat if:
\[
0 \to F' \to F \text{ exact} \iff 0 \to E \otimes F' \to E \otimes F \text{ exact}
\]
\begin{enumerate}
  
  (a) [5 points] Prove: \( E \) is \( F \)-flat \( \iff \) \( E \) is \( G \)-flat for every submodule \( G \) of \( F \). [Use Snake Lemma]

  (b) [5 points] Prove: \( E \) is \( F \)-flat \( \Rightarrow \) \( E \) is \( (F/G) \)-flat for every submodule \( G \) of \( F \). [Use Snake Lemma]

  (c) [5 points] Assume \( F = F_1 \oplus F_2 \). Prove: \( E \) is \( F \)-flat (\( i=1, 2 \)) \( \iff \) \( E \) is \( F \)-flat. [Use Snake Lemma]

  (d) [5 points] Assume \( F = \bigoplus_{i \in \Delta} F_i \). Prove: \( E \) is \( F \)-flat (\( \forall i \in \Delta \)) \( \iff \) \( E \) is \( F \)-flat. [Use (c)]

  (e) [5 points] Prove: \( E \) is flat \( \iff \) \( E \) is \( R \)-flat. [Use (d) and (b)]