King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 201: First Major Exam, Summer 153 (120 minutes)

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Name: **Student ID:** **Serial Number:**

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Solve all problems. Show **full details** of your solution.

Question		Grade
1		/12
2		/18
3		/10
4		/6
5		/15
6		/11
7		/8
8		/10
9		/10
TOTAL		

Q1. Consider the parametric curve

$$x = -\sin t,\ y = \cos t,\ \frac{\pi}{4} \le t \le \frac{3\pi}{2}$$

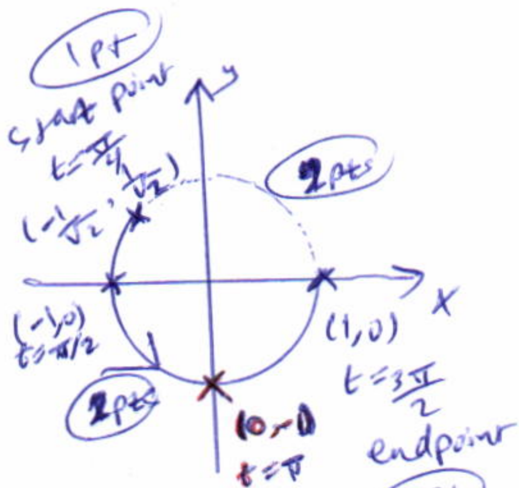
(a) (4 points) Find a cartesian equation for the curve.

$$x^2 + y^2 = (-\sin t)^2 + (\cos t)^2$$
$$= \sin^2 t + \cos^2 t = 1$$
$$\boxed{x^2 + y^2 = 1}$$

(b) (8 points) Sketch the curve and indicate the direction in which the curve is traced as t increases (with an arrow) as well as the *start* and the *end* points.

2 pts

t	x	y
$\frac{\pi}{4}$	$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{2}$	-1	0
π	0	-1
$\frac{3\pi}{2}$	1	0



Q2. Consider the parametric curve C

$$x = t^2, y = t - t^3, t \in \mathbb{R}.$$

(a) **(8 points)** Find the equations of the tangents to the curve C at the point $(x, y) = (1, 0)$.

1 pt $\frac{dx}{dt} = 2t$;

1 pt $\frac{dy}{dt} = 1 - 3t^2$.

3 pts $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{2t}$

$x = 1$ 1 pt
$t^2 = 1$
$t = \mp 1$
$y = 0$.

$m\big|_{t=-1} = \frac{dy}{dx}\Big|_{t=-1} = \frac{-2}{-2} = 1$

$y - 0 = 1(x-1)$ 1 pt

$\boxed{y = x - 1}$

$m\big|_{t=1} = \frac{dy}{dx}\Big|_{t=1} = \frac{-2}{2} = -1$ 1 pt

$y - 0 = -1(x-1)$; $\boxed{y = 1 - x}$

(b) **(6 points)** Find the point(s) at which the tangent is horizontal.

horizontal tangent when $\frac{dy}{dt} = 0$ & $\frac{dx}{dt} \neq 0$. 2 pts

$\frac{dy}{dt} = 0$

$1 - 3t^2 = 0$; $t = \mp \frac{1}{\sqrt{3}}$ 1 pt $\frac{dx}{dt}\Big|_{\frac{1}{\sqrt{3}}} = \frac{2}{\sqrt{3}} \neq 0$ & $\frac{dx}{dt}\Big|_{-\frac{1}{\sqrt{3}}} = \frac{-2}{\sqrt{3}} \neq 0$. 1 pt

So, h. tangent at $\left(\frac{1}{3}, \frac{2}{3\sqrt{3}}\right)$ & $\left(\frac{1}{3}, \frac{-2}{3\sqrt{3}}\right)$

1 pt 1 pt

(c) **(4 points)** Find the point(s) at which the tangent is vertical.

1 pt C has a vertical tangent when $\frac{dx}{dt} = 0$ & $\frac{dy}{dt} \neq 0$

$\frac{dx}{dt} = 0$

$2t = 0$; $\boxed{t = 0}$ 1 pt $\frac{dy}{dt}\Big|_{t=0} = 1 \neq 0$. 1 pt

So, c has a v. tangent at $\boxed{(x,y) = (0,0)}$.

Q3. (10 points) Find the length of the polar curve

$$r = 1 - \cos\theta$$

$\frac{dr}{d\theta} = \sin\theta$ (1 pt)

$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1-\cos\theta)^2 + (\sin\theta)^2$

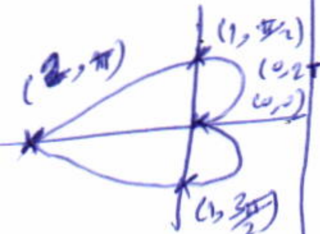
$= 1 - 2\cos\theta + (\cos^2\theta + \sin^2\theta)$

$= 2(1-\cos\theta)$ (2 pts)

$= 2(1 - \cos 2\cdot\frac{\theta}{2})$

$= 2(1 - \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2})$

$= 4\sin^2\frac{\theta}{2}$ (2 pts)



$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ (1 pt)

$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$

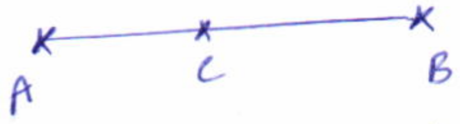
$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$ (1 pt)

$= \int_0^{2\pi} 2\cdot\sin\frac{\theta}{2}\, d\theta$

$= \left(-4\cos\frac{\theta}{2}\right)\Big]_0^{\pi}$

$= -4(-1-1) = 8$ (3 pts)

Q4 (6 points) Consider $A(1,2,5)$ and $B(0,1,3)$. Find the coordinates of the point C on the line segment AB such that $\left|\overrightarrow{AC}\right| = \frac{2}{3}\left|\overrightarrow{CB}\right|$.



Consider $C(a,b,c)$ on AB such that $|\vec{AC}| = \frac{2}{3}|\vec{CB}|$

As vectors $\boxed{\vec{AC} = \frac{2}{3}\vec{CB}}$ (2 pts)

$\langle a-1, b-2, c-5\rangle = \frac{2}{3}\langle 0-a, 1-b, 3-c\rangle$

$$\boxed{\begin{aligned} a-1 &= -\tfrac{2}{3}a \\ b-2 &= \tfrac{2}{3}(1-b) \\ c-5 &= \tfrac{2}{3}(3-c) \end{aligned}}$$ (2 pts)

$\frac{5}{3}a = 1$, $\frac{5}{3}b = \frac{8}{3}$, $\frac{5}{3}c = 7$

$a = \frac{3}{5}$, $b = \frac{8}{5}$, $c = \frac{21}{5}$

$C\left(\frac{3}{5}, \frac{8}{5}, \frac{21}{5}\right)$ (2 pts)

Q5. (15 Points) Use polar coordinates to find the area of the region common to the circles

$$x^2+y^2=4 \text{ and } x^2+y^2=4x$$

$r^2=4$ $\quad$ $r^2=4r\cos\theta$

$\boxed{r=2}$ $\quad$ $\boxed{r=4\cos\theta}$

$r_1=r_2$

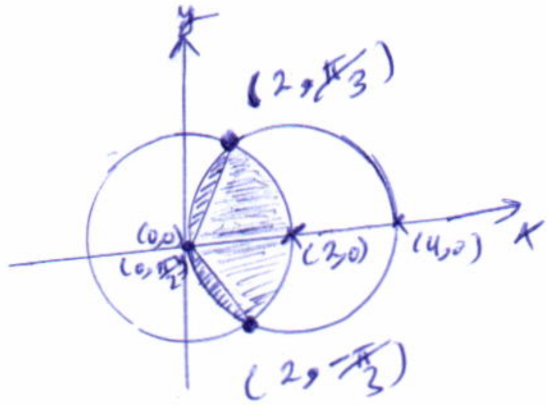
$2=4\cos\theta$

$\cos\theta=\frac{1}{2}$

$\theta=\frac{\pi}{3}+2n\pi$

or $\theta=-\frac{\pi}{3}+2n\pi$

2 pts



3 pts

$A=\frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$ 2 pts

$=2(A_1+A_2)$

$=2\left(\underbrace{\boxed{\frac{1}{2}\int_0^{\pi/3} 2^2\,d\theta}}_{A_1}+\underbrace{\boxed{\frac{1}{2}\int_{\pi/3}^{\pi/2}(4\cos\theta)^2\,d\theta}}_{A_2}\right)$ 2 pts 2 pts

1 pt

$\boxed{A_1=\frac{2\pi}{3}}$

$A_2=8\int_{\pi/3}^{\pi/2}\cos^2\theta\,d\theta=8\int_{\pi/3}^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$ 2 pts

$=4\left(\left(\theta+\frac{\sin 2\theta}{2}\right)\Big|_{\pi/3}^{\pi/2}\right)=4\left(\frac{\pi}{2}+0-\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)$

$=4\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)$

$=\boxed{\frac{2\pi}{3}-\sqrt{3}}$

$A=2(A_1+A_2)$

$\therefore A=2\left(\frac{2\pi}{3}+\frac{2\pi}{3}-\sqrt{3}\right)$

$\boxed{A=\frac{8\pi}{3}-2\sqrt{3}}$ 1 pt

Q6. Let $\overrightarrow{v} = < 2, -3, 1 >$ and $\overrightarrow{w} = < -1, 1, 4 >$.

(a) (5 points) Find two *unit* vectors parallel to the vector $2\overrightarrow{v} - 3\overrightarrow{w}$.

$2\vec{v} - 3\vec{w} = 2<2,-3,1> - 3<-1,1,4>$

$= <7,-9,-10>$ (2pts) ; $|2\vec{v} - 3\vec{w}| = \sqrt{49+81+100}$

$= \sqrt{230}$ (1pt)

(1pt) $\vec{u_1} = \frac{1}{\sqrt{230}} <7,-9,-10>$; $\vec{u_2} = -\vec{u_1}$ (1pt) $= \frac{1}{\sqrt{230}} <-7,9,10>$.

(b) (5 points) Find the vector projection of $\overrightarrow{v}$ on $\overrightarrow{w}$.

$\text{proj}_{\vec{w}} \vec{v} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$ (2pts) $= \frac{-2-3+4}{1+1+16} <-1,1,4>$ (1pt) (1pt)

$= \frac{-1}{18} <-1,1,4>$

$= <\frac{1}{18}, \frac{-1}{18}, \frac{-4}{18}>$. (1pt)

Q7 (8 points) Let $\overrightarrow{r} = < x, y, z >$, $\overrightarrow{v} = < 1, 2, 3 >$ and $\overrightarrow{w} = < 2, -1, 1 >$. Show that the vector equation

$$(\overrightarrow{r} - \overrightarrow{v}) \bullet (\overrightarrow{r} - \overrightarrow{w}) = 0$$

represents a sphere. Find the center and the radius of that sphere.

$\vec{r} - \vec{v} = <x-1, y-2, z-3>$ (1pt)

$\vec{r} - \vec{w} = <x-2, y+1, z-1>$ (1pt)

$(\vec{r} - \vec{v}) \cdot (\vec{r} - \vec{w}) = 0$

$(x-1)(x-2) + (y-2)(y+1) + (z-3)(z-1) = 0$ (3pts)

$x^2 - 3x + \cancel{2} + y^2 - y - \cancel{2} + z^2 - 4z + 3 = 0$.

$(x - \frac{3}{2})^2 + (y - \frac{1}{2})^2 + (z-2)^2 = \frac{9}{4} + \frac{1}{4} + 4 - 3$

$\boxed{(x - \frac{3}{2})^2 + (y - \frac{1}{2})^2 + (z-2)^2 = \frac{14}{4}}$ (1pt)

Sphere with center $(\frac{3}{2}, \frac{1}{2}, 2)$ (1pt) & radius $\frac{\sqrt{14}}{2}$. (1pt)

Q8. (10 points) Check whether the four points $P(3,0,1)$, $Q(-1,2,5)$, $R(5,1,-1)$ and $S(0,4,2)$ lie on the same plane or not.

$\overrightarrow{PQ} = \langle -4, 2, 4 \rangle$ 1 pt

$\overrightarrow{PR} = \langle 2, 1, -2 \rangle$ 1 pt

$\overrightarrow{PS} = \langle -3, 4, 1 \rangle$ 1 pt

$$\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) = \begin{vmatrix} -4 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$$ 2 pts 2 pts

$$= -4(1-(-8)) - 2(2-6) + 4(8-(-3))$$

$$= -36 + 8 + 44$$

$$= 16 \neq 0$$ 1 pt

So, the four points are not in the same plane. 2 pts