EXAM I
Wednesday, July 27, 2016

Allowed Time: 3 Hours

Instructions:
1. Write neatly and legibly -- you may lose points for messy work.
2. Show all your work -- no points for answers without justification.
3. Calculators and Mobiles are not allowed.
4. Make sure that you have 7 questions (8 pages + cover page + formula sheet).

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Coordinator: Dr A. N. Duman
Q1. If \( \mathbf{E} = 6r^2 \sin \theta \cos \phi \mathbf{\hat{r}} + 4r \cos 2\theta \sin(\phi/2) \mathbf{\hat{\theta}} + r^3 \mathbf{\hat{\phi}} \) at \( A(2, \frac{\pi}{6}, \frac{\pi}{3}) \) determine the vector component of \( \mathbf{E} \) that is:

(a) Tangential to the spherical surface \( r = 2 \). [5 pts]

(b) Normal to the surface \( \phi = \pi/3 \). [5 pts]

(c) Parallel to the line \( y = 2, z = 0 \). [5 pts]

(d) Determine a vector perpendicular to \( \mathbf{E} \) and tangential to the plane \( z = 2 \). [5 pts]
Q2.

(a) Find the magnitude and the direction of the maximum rate of change of
\[ T = \frac{2}{r} \sin \theta \cos \varphi \] at the point \( P(1, \frac{\pi}{6}, \frac{\pi}{2}) \). [5 pts]

(b) Find \( \nabla^2 V \) where is \( V = \rho^2 z \sin 3\varphi \) in cylindrical coordinates where \( \rho \neq 0 \). [5 pts]
Q3. Express $G = \frac{x \cos \varphi}{\rho} \hat{a}_x + \frac{2yz}{\rho^2} \hat{a}_y + (1 - \frac{x^2}{\rho^2}) \hat{a}_z$ completely in spherical system. [10 pts]
Q4. If \( \mathbf{E} = 2xyz \mathbf{\hat{x}} + x^2z \mathbf{\hat{y}} + x^2y \mathbf{\hat{z}} \), show that the line integral \( L \) between two arbitrary points \( A \) and \( B \) in a simply connected region containing the path,

\[
L: \int_{A}^{B} \mathbf{E} \cdot d\mathbf{l}
\]

is independent of the path. Use the potential function to evaluate \( L \) where, \( A = (2, 1, -1) \), and \( B = (5, 1, 2) \).

\([15 \text{ pts}]\)
Q5. Verify the divergence theorem for the function \( E = \rho^2 \hat{a}_\rho + z \cos \varphi \, \hat{a}_\varphi - \rho \sin \varphi \, \hat{a}_z \),
over cylinder defined by \( 1 < \rho < 3, \ 0 < z < 4 \). [20 points]
Q6. Using the curl of \( \mathbf{E} = r \sin \theta \, \hat{a}_r + \cos \phi \, \hat{a}_\theta - r^2 \cos \theta \, \hat{a}_\phi \), evaluate the circulation, 
\[
\int_C \mathbf{E} \cdot d\mathbf{l}
\]
around the closed path \( C \) which is the boundary of the open surface defined by 
\[0 < \theta < \pi/2, \ 0 < \phi < \frac{\pi}{4}, \ r = 2.\] 

[15 points]
Q7. Let \( E = 2r \sin \theta \cos \phi \, \hat{a}_r + r \cos \theta \cos \phi \, \hat{a}_\theta - r \sin \phi \, \hat{a}_\phi \) be the electric field on a certain region of space. Given two points \( A(1,0,0) \) and \( B(4, \pi/6, \pi/3) \) inside this region, find the electric potential at \( A \), i.e. \( V(A) \), given that \( V(B) = 5 \). \[10 \text{ points}\]
Formulae in cylindrical and spherical coordinate systems

Differential of displacement

Cylindrical: \( dl = d\rho \hat{\rho} + \rho d\phi \hat{\phi} + dz \hat{z} \)

Spherical: \( dl = dr \hat{r} + rd\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi} \)

Gradient of a scalar field, \( \nabla V \)

Cylindrical: \( \nabla V = \frac{\partial V}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{\phi} + \frac{\partial V}{\partial z} \hat{z} \)

Spherical: \( \nabla V = \frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\phi} \)

Divergence of a vector field, \( \nabla \cdot \mathbf{G} \)

Cylindrical: \( \nabla \cdot \mathbf{G} = \frac{1}{\rho} \frac{\partial (\rho G_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial G_\phi}{\partial \phi} + \frac{\partial G_z}{\partial z} \)

Spherical: \( \nabla \cdot \mathbf{G} = \frac{1}{r^2} \frac{\partial (r^2 G_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta G_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial G_\phi}{\partial \phi} \)

Relationship between Cartesian, Cylindrical and Spherical Coordinates

\[
\begin{pmatrix}
A_\rho \\
A_\phi \\
A_z
\end{pmatrix} = \begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
A_x \\
A_y \\
A_z
\end{pmatrix}
\]

\[
\begin{pmatrix}
A_r \\
A_\theta \\
A_\phi
\end{pmatrix} = \begin{pmatrix}
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\
\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0
\end{pmatrix} \begin{pmatrix}
A_x \\
A_y \\
A_z
\end{pmatrix}
\]