MATH 102.1 (Term 161)

Quiz 5 (Sects. 11.1, 11.2 & 11.3) Duration: 20min

Name: **ID number:**

1.)(2pts)Find the limit of the sequence $\left\{n\left(\sqrt{n^2+1}-n\right)\right\}_{n=1}^{\infty}$.

2.)(4pts)Evaluate the sums $\sum_{n=1}^{\infty}\frac{2^{2n-1}}{7^{n+1}}$, $\sum_{n=3}^{\infty}\frac{1}{n^2-n-2}$.

3.)(4pts)Do the series converge or diverge $\sum_{n=1}^{\infty}\frac{e^n}{e^{2n}+1}$, $\sum_{n=2}^{\infty}\left(\frac{1}{n^2}+\frac{n^3}{n^4+1}\right)$?

1.) $n(\sqrt{n^2+1}-n) = n\,\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$

$= \frac{1}{\sqrt{1+\frac{1}{n^2}}+1}$

$\lim_{n\to\infty} n(\sqrt{n^2+1}-n) = \frac{1}{2}$

2.) $\sum_{n=1}^{\infty}\frac{2^{2n-1}}{7^{n+1}} = \sum_{n=1}^{\infty}\frac{1}{14}\left(\frac{4}{7}\right)^n = \sum_{n=1}^{\infty}\frac{1}{14}\,\frac{4}{7}\left(\frac{4}{7}\right)^{n-1}$

$= \frac{\frac{1}{14}\cdot\frac{4}{7}}{1-\frac{4}{7}} = \frac{\frac{2}{7.7}}{3/7}$

$= \frac{2}{21}$

$a_n = \frac{3}{n^2-n-2} = \frac{3}{(n+1)(n-2)} = \frac{1}{n-2} - \frac{1}{n+1}$

$S_n = a_3 + a_4 + a_5 + \dots + a_n$

$= \left(1-\cancel{\frac{1}{4}}\right) + \left(\frac{1}{2}-\cancel{\frac{1}{5}}\right) + \left(\frac{1}{3}-\cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{7}}\right) + \dots +$

$+ \left(\cancel{\frac{1}{n-5}}-\cancel{\frac{1}{n-2}}\right) + \left(\cancel{\frac{1}{n-4}}-\frac{1}{n-1}\right) + \left(\cancel{\frac{1}{n-3}}-\frac{1}{n}\right) + \left(\cancel{\frac{1}{n-2}}-\frac{1}{n+1}\right)$

$= 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1}$

$\lim_{n\to\infty} S_n = \frac{11}{6} \Rightarrow \boxed{\sum_{n=3}^{\infty}\frac{1}{n^2-n-n} = \frac{11}{18}}$

3.) $f(x) = \frac{e^x}{e^{2x}+1}$ is positive

$f'(x) = \frac{e^x(1-e^{2x})}{(e^{2x}+1)^2} < 0,\ x>0$

From integral test, the serie $\sum\frac{e^n}{e^{2n}+1}$ and the integral $\int_1^{\infty} f(x)\,dx$ have the same nature.

Now, $\int_1^{\infty}\frac{e^x}{e^{2x}+1}\,dx = \lim_{b\to\infty}\left[\tan^{-1}(e^x)\right]_1^b$

$= \pi/2 - \tan^{-1}(e)$

$\Rightarrow \sum_{n=1}^{\infty}\frac{e^n}{e^{2n}+1}$ converges

---

b) $f(x) = \frac{x^3}{x^4+1}$, $f' = \frac{x^2(3-x^4)}{(x^4+1)^2} < 0,\ x>2$

$\int_1^{\infty}\frac{x^3}{x^4+1}\,dx = \lim_{b\to\infty}\frac{1}{4}\left[\ln(x^4+1)\right]_1^b = \infty$

$\sum\frac{n^3}{n^4+1}$ diverges by integral test

$\sum\frac{1}{n^2}$ converges

$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{1}{n^2}+\frac{n^3}{n^4+1}\right)$ diverges