Exercise 1. Let \( R = \{ (x, y) \in \mathbb{R}^2 : |x| = y^2 \} \)
Is \( R \) a function from \( \mathbb{R} \) to \( \mathbb{R} \)? why?

Solution.
For a given \( x \in \mathbb{R} \), \( x \) is related to the values \( \pm \sqrt{|x|} \). So, for example 1 is related to 1 and -1.
It follows that \( R \) is not a function.

Exercise 2. Let \( g : \mathbb{Q} \to \mathbb{Q} \) be the function defined by \( g(r) = 3r - 1 \), \( \forall r \in \mathbb{Q} \).

(i) Find \( g^{-1}(N) \), where \( N = \{1, 2, 3, \ldots \} \)

(ii) Find \( g^{-1}(E) \), where \( E \) is the set of all even integers.

Solution.

(i) \( g^{-1}(N) = \{ r \in \mathbb{Q} \mid g(r) \in N \} = \{ r \in \mathbb{Q} : 3r - 1 = n, \text{for some } n \in N \} \)

\[ = \{ r \in \mathbb{Q} \mid r = \frac{1+n}{3} \text{ for some } n \in N \} \]

\[ = \{ \frac{1+n}{3} \mid n \in N \} \]

(ii) \( g^{-1}(E) = \{ r \in \mathbb{Q} \mid g(r) \in E \} = \{ r \in \mathbb{Q} \mid 3r - 1 = 2n, \text{for some } n \in \mathbb{Z} \} \)

\[ = \{ \frac{1+2n}{3} \mid n \in \mathbb{Z} \} \]
Exercise 3.

1. Show that for all $a, b \in \mathbb{Z}$, we have

$$a \equiv b \pmod{7} \iff 5a + 2 \equiv 5b + 2 \pmod{7}$$

2. Deduce from (1) that $f : \mathbb{Z} \rightarrow \mathbb{Z}$

$$a \mapsto 5a + 2$$

is a well-defined injective function.

Is $f$ a bijection? Why?

**Solution.**

1. Assume $a \equiv b \pmod{7}$, then $a - b = 7k$, for some $k \in \mathbb{Z}$.

So $(5a + 2) - (5b + 2) = 5(a - b) = 5(7k)$; hence

$$7 \mid (5a + 2) - (5b + 2); \text{ that is to say } 5a + 2 \equiv 5b + 2 \pmod{7}.$$  

Conversely, assume $5a + 2 \equiv 5b + 2 \pmod{7}$; then

$$5a + 2 - (5b + 2)$$

is a multiple of 7.

Thus $5(a - b) = 7s$, for some $s \in \mathbb{Z}$.

As $7 \mid 5(a - b)$ and $\gcd(7, 5) = 1$, we have (by Gauss's Lemma) 7 divides $a - b$. Therefore $a \equiv b \pmod{7}$.

2. **Is $f$ well-defined?**

If $\overline{x} = \overline{y}$, then $\overline{5x + 2} = \overline{5y + 2} \ (\text{by (1)})$.

Hence $f$ is well-defined.

- **$f$ is injective.**

Let $\overline{x}, \overline{y} \in \mathbb{Z}$. If $f(\overline{x}) = f(\overline{y})$, then $\overline{5x + 2} = \overline{5y + 2}$; so by (2), $\overline{x} = \overline{y}$. Thus $f$ is one-to-one.
As for injective \( |f(\mathbb{Z}_7)| = |\mathbb{Z}_7| = 7 \), but as
\( f(\mathbb{Z}_7) \) is a subset of \( \mathbb{Z}_7 \) with the same cardinality, we have \( f(\mathbb{Z}_7) = \mathbb{Z}_7 \).

Therefore \( f \) is onto.

We conclude that \( f \) is a bijection.

**Exercise 4:** Find the inverse of the bijection

\[
\begin{align*}
f : \mathbb{R} \setminus \{4\} & \to \mathbb{R} \setminus \{3\} \\
x & \mapsto \frac{3x+1}{x-4}
\end{align*}
\]

**Solution.** Let \( y \in \mathbb{R} \setminus \{3\} \). Let us solve the equation

\[
f(x) = y, \quad \text{for} \quad x \in \mathbb{R} \setminus \{3\}.
\]

\[
f(x) = y \iff \frac{3x+1}{x-4} = y \iff 3x+1 = xy - 4y \iff 4y = 3x + 1
\]

\[
\iff x = \frac{4y + 1}{y - 3}
\]

So \( f \) is invertible and \( f^{-1} : \mathbb{R} \setminus \{3\} \to \mathbb{R} \setminus \{4\} \)

\[
y \mapsto f^{-1}(y) = \frac{4y + 1}{y - 3}
\]

**Exercise 5.** Let \( \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 3 & 6 & 5 & 4 & 1 & 2 \end{pmatrix} \)

\[\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 1 & 4 & 5 & 3 & 7 & 2 \end{pmatrix}\]

(i) Determine \( \alpha^{-1}, \beta^{-1}, \alpha^{-1} \beta^{-1}, \beta^{-1} \alpha^{-1} \)

(ii) Determine \( \alpha \beta, \beta \alpha \).
Solution.

(i)
\[ \alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 2 & 5 & 4 & 3 & 1 \end{pmatrix} \]
\[ \beta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 5 & 3 & 4 & 1 & 6 \end{pmatrix} \]
\[ \alpha^{-1} \beta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 2 & 5 & 4 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 5 & 3 & 4 & 1 & 6 \end{pmatrix} \]
\[ = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 1 & 4 & 2 & 5 & 6 & 3 \end{pmatrix} \]
\[ \beta^{-1} \alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 5 & 3 & 4 & 1 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 2 & 5 & 4 & 3 & 1 \end{pmatrix} \]
\[ = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 6 & 7 & 4 & 3 & 5 & 2 \end{pmatrix} \]

(ii)
\[ \alpha \beta = (\beta^{-1} \alpha^{-1})^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 7 & 5 & 4 & 6 & 2 & 3 \end{pmatrix} \]
\[ (\beta \alpha) = (\alpha^{-1} \beta^{-1})^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 7 & 3 & 5 & 6 & 1 \end{pmatrix} \]