EXAM II
December 7, 2016
Allowed Time: 150 mins

Student Name:
Student ID Number:
Section Number:
Instructor’s Name:

Instructions:
1. Write neatly and legibly -- you may lose points for messy work.
2. Show all your work -- no points for answers without justification.
3. Programmable calculators and Mobiles are not allowed.
4. Make sure that you have 4 questions (5 pages + cover page + formula sheet).

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<th>Problem No.</th>
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Coordinator: Dr A. N. Duman
Q1. Let $\mathbf{A} = 3z^2 \sin \varphi \hat{\rho} + \rho \cos 2\varphi \hat{\varphi} - \rho z \hat{z}$ at $P(2\sqrt{3}, \frac{\pi}{6}, 2)$.

(a) Determine the vector component of $\mathbf{A}$ that is tangential to the surface $\theta = \pi/3$. Give your answer in cylindrical coordinates. [15 pts]

(b) Determine the angle that $\mathbf{A}$ makes the tangent plane of the surface $r = 4$. [10 pts]
Q2.
(a) Find the directional derivative \( T = r^2 \sin \theta \cos \varphi \) in the direction \( 3\mathbf{a}_x - 4\mathbf{a}_z \) at the point \( P(1, \frac{\pi}{6}, \frac{\pi}{2}) \). \[10 \text{ pts}\]
(b) Find \( \nabla^2 V \) in cylindrical coordinates, where \( V = \rho z \cos 2\varphi \), \( \rho \neq 0 \). \[5 \text{ pts}\]
(c) Express \( \nabla V \) in spherical coordinates. \[10 \text{ pts}\]
Q3. Verify the divergence theorem for the function \( \mathbf{E} = 2\rho z^2 \mathbf{\hat{a}}_\rho + \rho \cos^2 \phi \mathbf{\hat{a}}_z \), over region defined by \( 2 < \rho < 5, -1 < z < 1, 0 < \phi < 2\pi \). \[30 \text{ points}\]
Q4. Let \( \mathbf{E} = (20\rho \sin \varphi + 6z)\mathbf{a}_\rho + 10 \rho \cos \varphi \mathbf{a}_\varphi + 6\rho \mathbf{a}_z \) be the electric field on a certain region of space.

(a) Verify that \( \mathbf{E} \) is a conservative field. [5 points]

(b) Find the electric potential function \( V \). [10 points]

(c) Given two points \( A(1,0,1) \) and \( B(4,\pi/6,0) \) inside this region, find the electric potential at \( A \), i.e. \( V(A) \), given that \( V(B) = 5 \). [5 points]
**Formulae in cylindrical and spherical coordinate systems**

**Differential of displacement**

Cylindrical:  
\[ dl = d\rho \, \hat{\rho} + \rho d\phi \, \hat{\phi} + dz \, \hat{z} \]

Spherical:  
\[ dl = dr \, \hat{r} + rd\theta \, \hat{\theta} + r \sin \theta \, d\phi \, \hat{\phi} \]

**Gradient of a scalar field, \( \nabla V \)**

Cylindrical:  
\[ \nabla V = \frac{\partial V}{\partial \rho} \, \hat{\rho} + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \, \hat{\phi} + \frac{\partial V}{\partial z} \, \hat{z} \]

Spherical:  
\[ \nabla V = \frac{\partial V}{\partial r} \, \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \, \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \, \hat{\phi} \]

**Divergence of a vector field, \( \nabla \cdot \mathbf{G} \)**

Cylindrical:  
\[ \nabla \cdot \mathbf{G} = \frac{1}{\rho} \frac{\partial (\rho G_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial G_\phi}{\partial \phi} + \frac{\partial G_z}{\partial z} \]

Spherical:  
\[ \nabla \cdot \mathbf{G} = \frac{1}{r^2} \frac{\partial (r^2 G_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta G_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial G_\phi}{\partial \phi} \]

**Relationship between Cartesian, Cylindrical and Spherical Coordinates**

\[
\begin{pmatrix}
A_\rho \\
A_\phi \\
A_z
\end{pmatrix} =
\begin{pmatrix}
\cos \varphi & \sin \varphi & 0 \\
-\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
A_x \\
A_y \\
A_z
\end{pmatrix}
\]

\[
\begin{pmatrix}
A_r \\
A_\theta \\
A_\phi
\end{pmatrix} =
\begin{pmatrix}
\sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \\
\cos \theta \cos \varphi & \cos \theta \sin \varphi & -\sin \theta \\
-\sin \varphi & \cos \varphi & 0
\end{pmatrix}
\begin{pmatrix}
A_x \\
A_y \\
A_z
\end{pmatrix}
\]