King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 513 Final Exam
The First Semester of 2016-2017 (161)

Time Allowed: 180 Minutes

Name: ___________________________ ID#: ______________
Section/Instructor: _________________ Serial #: ______________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

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Q: 1 (15 points) Solve the Sturm-Liouville problem:

\[ y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(\pi) + y'(\pi) = 0. \]
Q:2 (15 points) Solve the heat equation
\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \quad t > 0 \]
subject to the following initial and non-homogeneous boundary conditions
\[ u(x, 0) = 1, \quad u(0, t) = T_0, \quad u_x(\pi, t) = 0, \quad 0 < x < \pi, \quad t > 0, \]
where $T_0$ is any constant.
Q:3 (15 points) Use Laplace transform method to solve the wave equation

\[ \frac{\partial^2 u}{\partial x^2} + \sin \pi x \sin \omega t = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \quad t > 0 \]

with the boundary and initial conditions

\[ u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0 \]

\[ u(x, 0) = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0, \quad 0 < x < 1. \]
Q: 4 (20 points) Solve the Laplace equation by separation of variables

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1, \quad 0 < y < 1, \]
\[ u(x, 0) = 0, \quad u(0, y) = 10y, \]
\[ \left. \frac{\partial u}{\partial x} \right|_{x=1} = -1, \quad u(x, 1) = 0. \]
Q:5 (20 points) Solve

\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < 1, z > 0 \]

subject to the following boundary conditions

1. \[ u_r|_{r=1} + hu(1, z) = 0, \quad z > 0, \]
2. \[ u(r, 0) = 4, \quad 0 < r < 1. \]
Q:6 (20 points) Find the steady-state temperature in the sphere of radius $C$ by solving

\[
\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0, \quad 0 < r < C, \quad 0 < \theta < \pi
\]

\[
u(C, \theta) = \cos(\theta), \quad 0 < \theta < \pi.
\]
Q:7 (10 points) Let \( A = \begin{pmatrix} -1 & 2 & -2 \\ 2 & -1 & 2 \\ -2 & 2 & -1 \end{pmatrix} \)

(a) Find the eigenvalues and eigenvectors of \( A \)
(b) Find an orthogonal matrix $P$ that diagonalizes $A$ and find the diagonal matrix $D = P^T AP$. 