King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH533 - Complex Variables
Final Exam – Semester 161

Exercise 1

Use residues to evaluate the following definite integral

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin(\theta)}$$

Exercise 2

Evaluate

(a) $\displaystyle\int_{|z|=1} \frac{e^z}{z(2z+1)^2}\,dz$

(b) $\displaystyle\int_{|z|=2} (2z-1)e^{\frac{z}{z-1}}\,dz$ (Hint: use residue at ∞)

Exercise 3

True or False (if true, give a short explanation, if false, give a counterexample)

(a) Let f be an entire function such that $\displaystyle\int_{|z|=R} \frac{f'(z)}{f(z)} dz = 0$ for $R > 2017$. Then f is constant.

(b) If f has a removable singularity at ∞, then $Res(f, \infty) = 0$.

(c) If f has a removable singularity at $z_0 \in \mathbb{C}$, then $Res(f, z_0) = 0$.

(d) If two entire functions agree on a segment of the real axis, then they agree on $\mathbb{C}$.

Exercise 4

Let $\Omega := \{z \in \mathbb{C} : |z| > 1\}$.

- Let $f(z) = \dfrac{z}{z-1}$. Show that $\dfrac{f'(z)}{f(z)}$ has an antiderivative on Ω.
- Show that $\dfrac{z}{z-1}$ has an analytic logaritm on Ω. You may find the result in part (a) useful.

Exercise 5

(a) Show that the transformation $w = \dfrac{z-1}{z+1}$ maps the half-plane $\{z \in \mathbb{C} : \Re(z) > 0\}$ onto $|w| < 1$.

(b) Suppose that f is analytic on the half-plane $\{z \in \mathbb{C} : \Re(z) > 0\}$ and $|f(z)| \leq 1$. Show that $|f(2)| \leq 1/3$ if $f(1) = 0$.

Exercise 6

Find all functions $f(z)$ which have in the extended complex plane only the following singularities: a pole of order 3 at $z = 0$ and a pole of order 2 at $z = \infty$.

Exercise 7

Let $f : \Delta \to \Delta$ be analytic from the unit disc to the unit disk.

1. Show that $|f^{(n)}(0)| \leq n!$
2. Prove if $f(0) = f'(0) = 0$ then $|f(z)| \leq |z|^2$ and $|f''(0)| \leq 2$.
3. Find all analytic functions $f : \Delta \to \Delta$ such that $f(0) = f'(0) = 0$ and $|f''(0)| = 2$.

Exercise 8

(a) Show that the equation $z^5 + 15z + 1 = 0$ has precisely four solutions in the annulus $\{z \in \mathbb{C} : 3/2 < |z| < 2\}$.

(b) Let f be analytic in a neighborhood of $\overline{\Delta}$. If $|f(z)| < 1$ for $|z| = 1$, show that there is a unique z with $|z| < 1$ and $f(z) = z$. If $|f(z)| \leq 1$ for $|z| = 1$, what can you say?