King Fahd University Petroleum and Minerals
Department of Mathematics and Statistics

MASTER | **MATH 201 - Term 162 - Final Exam** | MASTER

Duration: 180 minutes

KEY

Name:__________ ID Number:__________

Section Number:__________ Serial Number:__________

ClassTime: __________ Instructor's Name:__________

General Instructions:

1. Calculators and Mobiles are not allowed.
2. This exam consists of two parts: Written and Multiple Choice.

Parts	Points	Maximum Points
Written		70
MCQ		70
Total		140

Part I: Written Questions

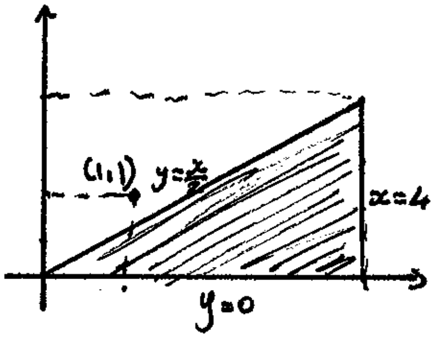
Instructions for Written Questions

1. This part has 6 written questions.
2. Answer the questions in the space provided.
3. Show your work. Points will be deducted for results without work.
4. Write clearly. Points may be deducted for poor presentation.
5. No credits will be given to wrong steps.

Question Number	Points	Maximum Points
1		15
2		14
3		12
4		12
5		8
6		9
Total		70

1. (15-points) Find the absolute maximum and minimum values of $f(x,y) = x^2 + xy - y^2 - 3x + y$ on the triangular region with vertices $(0,0)$, $(4,0)$, and $(4,2)$.

1pt $\begin{cases} f_x(x,y) = 2x+y-3=0 \\ f_y(x,y) = x-2y+1=0 \end{cases} \Rightarrow \begin{cases} 2x+y=3 \\ x-2y=-1 \end{cases} \Rightarrow (x,y)=(1,1)$ 1pt



1pt As $(1,1)$ is not in the triangular region it is not a critical point.

On the boundary

3pts
* $y=0$, $g(x) = f(x,0) = x^2-3x$ on $[0,4]$ $g'(x) = 2x-3=0 \Rightarrow x=\frac{3}{2}$
As $x=\frac{3}{2}$ is in $[0,4]$, the point $(\frac{3}{2},0)$ is a critical point of $g(x)$.
the points of interest are $(\frac{3}{2},0)$, $(0,0)$, $(0,4)$

3pts
* $x=4$: $h(y) = f(4,y) = -y^2+5y+4$ on $[0,2]$ $h'(y) = -2y+5=0 \Rightarrow y=\frac{5}{2}$
As $y=\frac{5}{2}$ is not in $[0,2]$, $(\frac{1}{4},\frac{5}{2})$ is not a critical point of $h(y)$.
the points of interest are $(4,0)$, $(4,2)$

3pts
* $y=\frac{x}{2}$: $k(x) = f(x,\frac{x}{2}) = \frac{5x^2}{4} - \frac{5x}{2}$ on $[0,4]$ $k'(x) = \frac{5x}{2} - \frac{5}{2} \Rightarrow x=1$.
As $x=1$ is in $[0,4)$, $(1,\frac{1}{2})$ is a critical point of $k(x)$.
the points of interest are $(0,0)$ $(1,\frac{1}{2})$ $(4,2)$

1pt $f(0,0)=0$ $f(4,0)=4$ $f(4,2)=10$ $f(\frac{3}{2},0) = \frac{-9}{4}$ $f(1,\frac{1}{2}) = \frac{-5}{4}$

Conclusion

1pt * the absolute maximum value is 10

1pt * the absolute minimum value is $-\frac{9}{4}$

2. (14-points) Use **Lagrange Multipliers** method to find the maximum and minimum values of $f(x, y, z) = xy + xz$ subject to the constraint $x^2 + y^2 + z^2 = 4$.

1pt $\nabla f(x,y,z) = \langle y+z, x, x \rangle$

1pt $\nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle$ where $g(x,y,z) = x^2+y^2+z^2-4$

2pts We solve the system $\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ x^2+y^2+z^2=4 \end{cases} \Rightarrow \begin{cases} y+z = 2\lambda x & (1) \\ x = 2\lambda y & (2) \\ x = 2\lambda z & (3) \\ x^2+y^2+z^2=4 & (4) \end{cases}$

2pts If $x=0$, then $y=-z$ from (1). From (4), $2y^2=4 \Rightarrow y = \pm\sqrt{2}$.
We find points $(0, \sqrt{2}, -\sqrt{2})$ and $(0, -\sqrt{2}, \sqrt{2})$.

1pt If $x \neq 0$, then $y \neq 0$, $z \neq 0$ and $\lambda \neq 0$.
From (2) and (3), $y = z$. From (1), $y = \lambda x$. Using (2), $\lambda = \pm\frac{1}{\sqrt{2}}$

2pts If $\lambda = \frac{1}{\sqrt{2}}$, then $z = y = \frac{x}{\sqrt{2}}$. From (4), $x = \pm\sqrt{2}$
if $x = \sqrt{2}$, then $y = z = 1$ and we find $(\sqrt{2}, 1, 1)$.
if $x = -\sqrt{2}$, then $y = z = -1$ and we find $(-\sqrt{2}, -1, -1)$

2pts If $\lambda = -\frac{1}{\sqrt{2}}$, then $z = y = \frac{-x}{\sqrt{2}}$ From (4), $x = \pm\sqrt{2}$
if $x = \sqrt{2}$ then $y = z = -1$ and we find $(\sqrt{2}, -1, -1)$.
if $x = -\sqrt{2}$ then $y = z = 1$ and we find $(-\sqrt{2}, 1, 1)$.

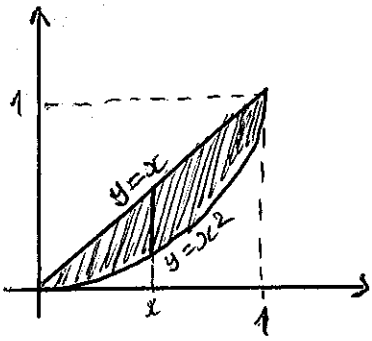
1pt
$f(0, \sqrt{2}, -\sqrt{2}) = f(0, -\sqrt{2}, \sqrt{2}) = 0$
$f(\sqrt{2}, 1, 1) = f(-\sqrt{2}, -1, -1) = 2\sqrt{2}$
$f(\sqrt{2}, -1, -1) = f(-\sqrt{2}, 1, 1) = -2\sqrt{2}$

Conclusion

1pt max. value is $2\sqrt{2}$

1pt min value is $-2\sqrt{2}$

3. (12-points) Evaluate $\displaystyle\iint_D \frac{x}{(1+y)^2}\, dA$ where D is the region in the 1^{st} quadrant enclosed by $y = x$ and $y = x^2$.



$$D = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le x\} \quad \underline{\underline{2\,\text{pts}}}$$

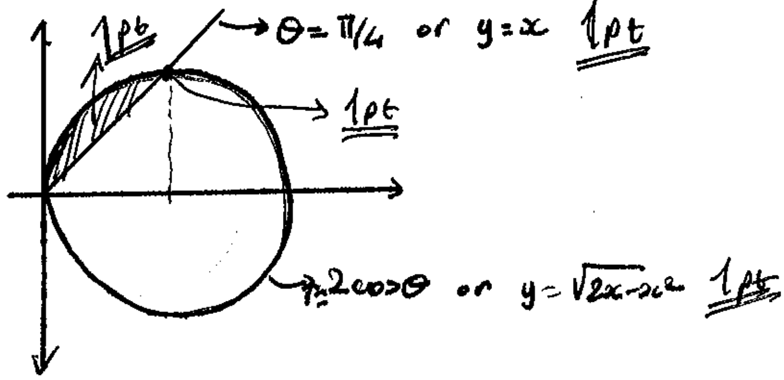
$$\iint_D \frac{x}{(1+y)^2}\, dA = \underbrace{\int_0^1\int_{x^2}^{x} \frac{x}{(1+y)^2}\, dy\, dx}_{2\,\text{pts}} = \underbrace{\int_0^1 \frac{-x}{1+y}\Big|_{x^2}^{x} dx}_{1\,\text{pt}}$$

$$= \underbrace{\int_0^1 \frac{x}{1+x^2} - \frac{x}{1+x}\, dx}_{1\,\text{pt}} = \underbrace{\frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\, dx}_{1\,\text{pt}} - \underbrace{\int_0^1 dx + \int_0^1 \frac{1}{1+x}\, dx}_{1\,\text{pt}}$$

$$= \underbrace{\frac{1}{2}\ln(1+x^2) - x + \ln(1+x)\Big|_0^1}_{3\,\text{pts}} = \frac{1}{2}\ln(2) - 1 + \ln(2) = \underbrace{\frac{3}{2}\ln(2) - 1}_{1\,\text{pt}}$$

4. Consider the double integral $\int_0^1 \int_x^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\; dydx$.

(a) (4-points) Sketch the region of integration.



(b) (8-points) Evaluate the double integral by changing into **polar coordinates**.

$$D = \{(r,\theta) \mid \pi/4 \le \theta \le \pi/2 \;,\; 0 \le r \le 2\cos\theta\}$$

$$\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r \; \underline{r\,dr\,d\theta} = \frac{1}{3}\int_{\pi/4}^{\pi/2} r^3\Big|_0^{2\cos\theta} d\theta = \underbrace{\frac{8}{3}\int_{\pi/4}^{\pi/2}\cos^3\theta\, d\theta}_{1pt}$$

1pt 1pt 1pt 1pt

$$= \frac{8}{3}\int_{\pi/4}^{\pi/2}(1-\sin^2\theta)\cos\theta\, d\theta = \underbrace{\frac{8}{3}\int_{1/\sqrt{2}}^{1} 1-u^2\, du}_{1pt} = \underbrace{\frac{8}{3}\left(u - \frac{u^3}{3}\right)\Big|_{1/\sqrt{2}}^{1}}_{1pt}$$

do substitution
$u = \sin\theta$ $du = \cos\theta d\theta$
$\theta = \pi/4 \Rightarrow u = 1/\sqrt{2}$
$\theta = \pi/2 \Rightarrow u = 1$

$$= \frac{8}{3}\left(\left(1 - \frac{1}{3}\right) - \left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right)\right) = \frac{8}{3}\left(\frac{2}{3} - \frac{5}{6\sqrt{2}}\right)$$

1pt

5. (8-points) **Use triple integrals in cylindrical coordinates** to find the volume of the solid enclosed by the paraboloid $z = 4 - x^2 - y^2$ and the plane $z = -5$.

The base of the solid is the circle

$$4 - x^2 - y^2 = -5 \Rightarrow x^2 + y^2 = 9.$$

The solid is described by

$$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 3, -5 \leq z \leq 4 - r^2\}. \quad \underline{3\text{pts}}$$

$$\text{Volume of } E = \underbrace{\iiint_E dV}_{1\text{pt}} = \underbrace{\int_0^{2\pi} \int_0^3 \int_{-5}^{4-r^2} r \, dz \, dr \, d\theta}_{1\text{pt}}$$

$$= 2\pi \int_0^3 rz \Big|_{-5}^{4-r^2} dr = \underbrace{2\pi \int_0^3 r(9 - r^2) dr}_{1\text{pt}}$$

$$= 2\pi \left(\frac{9}{2}r^2 - \frac{r^4}{4}\right)\Big|_0^3 \quad \underline{1\text{pt}}$$

$$= 2\pi \left(\frac{81}{2} - \frac{81}{4}\right)$$

$$= \frac{81\pi}{2} \quad \underline{1\text{pt}}$$

6. (9-points) Evaluate by changing into **spherical coordinates** the triple integral $\iiint_E \sin\left((x^2+y^2+z^2)^{3/2}\right)\, dV$ where E is the the unit ball $x^2+y^2+z^2 \le 1$.

$E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$. 3pts

$$\iiint_E \sin\left((x^2+y^2+z^2)^{3/2}\right) dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \underbrace{\sin(\rho^3)}_{1pt}\, \underbrace{\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi}_{1pt}$$

$$= \underbrace{\int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi \int_0^{1} \sin(\rho^3)\rho^2\, d\rho}_{1pt} = 2\pi \underbrace{\left(-\cos\phi\right)\Big|_0^{\pi}}_{1pt} \underbrace{\left(-\frac{1}{3}\cos(\rho^3)\right)\Big|_0^{1}}_{1pt}$$

$$= 2\pi\,(2)\,\frac{1}{3}\left(1-\cos(1)\right) = \frac{4\pi}{3}\left(1-\cos(1)\right) \quad 1pt$$

Part II: Multiple Choice Questions

Instructions for Multiple Choice Questions

1. This part has 14 multiple choice questions.
2. Each question carries 5 points.
3. No partial credit.
4. Transfer all your answers to the provided answer sheet.
5. Use HB 2.5 pencils only.
6. Use a good eraser. DO NOT use the erasers attached to the pencil.
7. Write your name, ID number and Section number on the examination paper and in the upper left corner of the answer sheet.
8. When bubbling your ID number and Section number, be sure that the bubbles match with the numbers that you write.
9. The Test Code Number is already bubbled in your answer sheet. Make sure that it is the same as that printed on your question paper.
10. When bubbling, make sure that the bubbled space is fully covered.
11. When erasing a bubble, make sure that you do not leave any trace of penciling.

1. Which one of the below points given in polar coordinates is in the region described by the polar inequalities $-1 \leq r \leq 3$ and $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$?

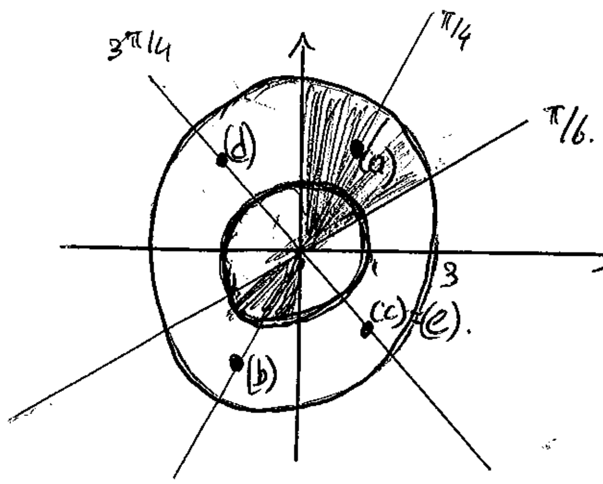
(a) $\left(-2, -\frac{3\pi}{4}\right)$

(b) $\left(-2, \frac{\pi}{4}\right)$

(c) $\left(2, -\frac{\pi}{4}\right)$

(d) $\left(2, \frac{3\pi}{4}\right)$

(e) $\left(-2, \frac{3\pi}{4}\right)$



* The region in question is shaded in the above picture

* The location of all points in the choices are marked on the above picture.

2. Let **a** and **b** be two unit vectors and let θ be the angle between them. If $\mathbf{a} + \text{proj}_{\mathbf{a}}\, \mathbf{b} = \mathbf{0}$, then θ is

(a) π

(b) 0

(c) $\pi/2$

(d) $\pi/4$

(e) $3\pi/4$

$$a + \frac{a \cdot b}{|a|^2} \cdot a = 0$$

$|a| = 1$

$$a + (a \cdot b) \cdot a = 0$$

$a \cdot b = |a|\,|b| \cos\theta = \cos\theta.$

$$a + \cos\theta \; a = 0$$

$$a\,(1 + \cos\theta) = 0 \Rightarrow 1 + \cos\theta = 0 \Rightarrow \theta = \pi.$$

3. If there exists a line through the points $P(2,3,1)$, $Q(3,a,-2)$, and $R(-1,9,b)$, then $a+b$ is equal to

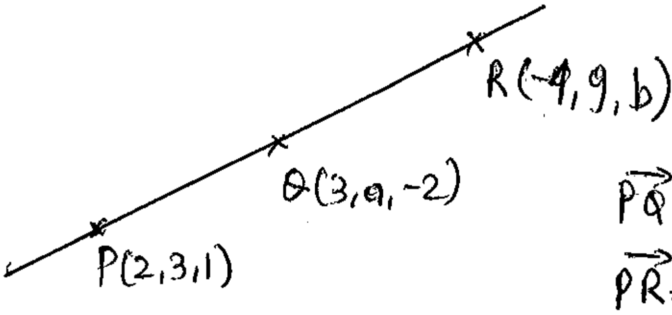
(a) 11

(b) 6

(c) 9

(d) -3

(e) -2



$\vec{PQ} = \langle 1, a-3, -3 \rangle$

$\vec{PR} = \langle -3, 6, b-1 \rangle$.

$\vec{PQ} \parallel \vec{PR} \Rightarrow \vec{PQ} \times \vec{PR} = \vec{0} \Rightarrow \begin{vmatrix} i & j & k \\ 1 & a-3 & -3 \\ -3 & 6 & b-1 \end{vmatrix} = \langle 0,0,0 \rangle$.

$\Rightarrow \langle (a-3)(b-1)+18,\ -(b-10),\ 6+3a-9 \rangle = \langle 0,0,0 \rangle$.

$\Rightarrow \begin{cases} (a-3)(b-1)+18=0 \\ b-10=0 \\ 3a-3=0 \end{cases} \Rightarrow$ From last two equations we find $a=1$ and $b=10$ so $a+b=11$.

Remark that $a=1$ $b=10$ also satisfies the 1st equation.

4. If $f(x,y) = (x^2+y)\, e^{\tan^{-1}\left(\frac{y}{x^2}\right)} - \sqrt{xy+1}\, \cos\left(\dfrac{y}{x^2}\right)$, then $f_x(1,0) =$

(Hint: You may use the limit definition of partial derivatives.)

(a) 2

(b) 1

(c) 0

(d) -1

(e) -2

$$f_x(1,0) = \lim_{x\to 1} \frac{f(x,0)-f(1,0)}{x-1}$$

$$= \lim_{x\to 1} \frac{(x^2-1)-0}{x-1}$$

$$= \lim_{x\to 1} x+1$$

$$= 2.$$

7. Suppose f is a function of x and y and $g(r,s) = f(2r-s, s^2-4r)$. Given the table,

(x,y)	f	f_x	f_y
$(1,-1)$	1	1	3
$(3,-3)$	2	5	4

$x = 2r-s \quad y = s^2-4r$

$x_r = 2 \quad y_r = -4$

$g_r(1,-1)$ is equal to

$$g_r(1,-1) = f_x(3,-3)x_r(1,-1) + f_y(3,-3)y_r(1,-1)$$
$$= 5 \cdot 2 + 4 \cdot (-4)$$
$$= -6$$

(a) -6

(b) 18

(c) -4

(d) -8

(e) -11

8. An equation of the tangent plane to the surface $z = \frac{1}{2}x^6y^{-2}$ at the point $(2,4,2)$ is

Verify that $(2,4,2)$ is on the surface:

$\frac{1}{2} 2^6 \cdot 4^{-2} = \frac{1}{2} 2^6 \cdot 2^{-4} = 2$

Consider $F(x,y,z) = \frac{1}{2}x^6y^{-2} - z$.

$\nabla F(x,y,z) = \langle 3x^5y^{-2}, -x^6y^{-3}, -1 \rangle$.

$\nabla F(2,4,2) = \langle 6, -1, -1 \rangle$.

Eqn of T.P

$6(x-2) - (y-4) - (z-2) = 0$

or

$6x - y - z = 6$.

(a) $6x - y - z = 6$

(b) $3x + y + z = 12$

(c) $6x + y + z = 18$

(d) $3x - y - z = 0$

(e) $x + 3y + 3z = 8$

9. Given that the point $(1,0)$ is a critical point of $f(x,y) = 3xe^y - x^3 - e^{3y}$ then which one of the following statement is **TRUE**?

(a) f has a local maximum at $(1,0)$.

(b) f has a local minimum at $(1,0)$.

(c) f has a saddle point at $(1,0)$.

(d) $D(1,0) = 0$.

(e) $f(1,0) = 0$.

$f_x(x,y) = 3e^y - 3x^2$

$f_{xx}(x,y) = -6x \Rightarrow f_{xx}(1,0) = -6$

$f_y(x,y) = 3xe^y - 3e^{3y}$

$f_{yy}(x,y) = 3xe^y - 9e^{3y} \Rightarrow f_{yy}(1,0) = -6 < 0$

$f_{xy}(x,y) = 3e^y \Rightarrow f_{xy}(1,0) = 3$

$D(1,0) = 36 - 9 = 25 > 0$

f has a local maximum at $(1,0)$.

10. If $R = [-2,2] \times [0,4]$, then using the Midpoint Rule with $m = n = 2$, the estimate of $\int\int_R \ln(x^2+y)\,dA$ is equal to

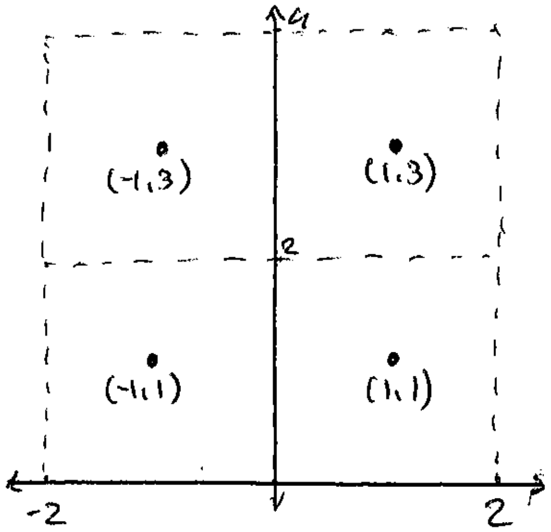
(a) $24 \ln 2$

(b) $8 \ln 2$

(c) $48 \ln 2$

(d) $16 \ln 2$

(e) $6 \ln 2$



Area of one square is 4.

Midpoints are $(-1,1), (1,1), (-1,3), (1,3)$

Estimate of the integral $= 4 \cdot (\ln 2 + \ln 2 + \ln 4 + \ln 4) = 24 \ln 2$.

11. If $R = [1,2]\times[0,1]$, then $\displaystyle\iint_R \frac{1}{\sqrt{1+x+y}}\,dA =$

(a) $\frac{8}{3}(4+\sqrt{2}-3\sqrt{3})$

(b) $\frac{1}{3}(8-2\sqrt{2}+\sqrt{3})$

(c) $\frac{3}{2}(4+\sqrt{2}-3\sqrt{3})$

(d) $\frac{3}{4}(4-2\sqrt{2}+3\sqrt{3})$

(e) $\frac{5}{4}(8+2\sqrt{2}-\sqrt{3})$

$$\int_1^2\int_0^1 \frac{1}{\sqrt{1+x+y}}\,dy\,dx$$

$$\int_1^2 2\sqrt{1+x+y}\,\Big|_0^1\,dx = 2\int_1^2 \sqrt{x+2}-\sqrt{x+1}\,dx$$

$$= 2\cdot\frac{2}{3}\left((x+2)^{3/2}-(x+1)^{3/2}\right)\Big|_1^2$$

$$= \frac{4}{3}\left(4^{3/2}-3^{3/2}-3^{3/2}+2^{3/2}\right)$$

$$= \frac{4}{3}\left(8-6\sqrt{3}+2\sqrt{2}\right)$$

$$= \frac{8}{3}\left(4+\sqrt{2}-3\sqrt{3}\right).$$

12. $\displaystyle\int_0^1\int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy =$

(Hint: Change the order of integration)

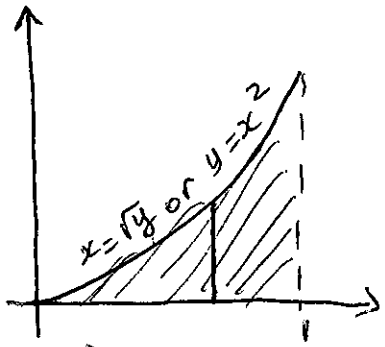
(a) $\frac{2}{9}(2\sqrt{2}-1)$

(b) $\frac{4}{9}(\sqrt{2}+2)$

(c) $\frac{2}{9}(\sqrt{2}+1)$

(d) $\frac{1}{9}(2\sqrt{2}+1)$

(e) $\frac{1}{9}(\sqrt{2}-2)$



$0\le x\le 1 \qquad 0\le y\le x^2.$

$$\int_0^1\int_0^{x^2}\sqrt{x^3+1}\,dy\,dx = \int_0^1 x^2\sqrt{x^3+1}\,dx = (x^3+1)^{3/2}\cdot\frac{2}{9}\Big|_0^1$$

$$= \frac{2}{9}\left(2\sqrt{2}-1\right)$$

13. The average value of $f(x,y) = \dfrac{2}{1+x^2+y^2}$ on the region $D = \{(x,y) | 1 \le x^2+y^2 \le 9\}$ is equal to

(a) $\frac{1}{4}\ln 5$

(b) $2\pi \ln 41$

(c) $2\pi \ln 80$

(d) $\frac{1}{4}\ln 8$

(e) $\frac{1}{4}\ln 12$

Area of D is $9\pi - \pi = 8\pi$.

Average value $= \dfrac{1}{8\pi}\iint_D \dfrac{2}{1+x^2+y^2}\, dA.$

$$= \frac{1}{8\pi}\int_0^{2\pi}\int_1^3 \frac{2}{1+r^2}\, r\, dr\, d\theta$$

$$= \frac{1}{8\pi} \cdot 2\pi \cdot \ln(1+r^2)\Big|_1^3$$

$$= \frac{1}{4}\left(\ln(10) - \ln(2)\right) = \frac{1}{4}\ln 5.$$

14. $\displaystyle\int_0^{\sqrt{2}}\int_0^{y}\int_0^{yz} y^2 e^x \, dx\, dz\, dy =$

(a) $\frac{1}{2}(e^2 - 5)$

(b) $\frac{1}{2}(e^2 + 3)$

(c) $\frac{e^2}{2}$

(d) $e^2 + \frac{5}{2}$

(e) $e^2 - \frac{3}{2}$

$$\int_0^{\sqrt{2}}\int_0^{y} y^2 e^x \Big|_0^{yz} dz\, dy = \int_0^{\sqrt{2}}\int_0^{y} y^2 e^{yz} - y^2\, dz\, dy.$$

$$= \int_0^{\sqrt{2}} y e^{yz} - y^2 z \Big|_0^{y} dy = \int_0^{\sqrt{2}} y e^{y^2} - y^3 - y\, dy$$

$$= \left(\frac{e^{y^2}}{2} - \frac{y^4}{4} - \frac{y^2}{2}\right)\Big|_0^{\sqrt{2}} = \left(\frac{e^2}{2} - 1 - 1\right) - \frac{1}{2} = \frac{e^2}{2} - \frac{5}{2}$$

Q	MM	V1	V2	V3	V4
1	a	a	b	d	a
2	a	d	a	e	e
3	a	b	a	d	b
4	a	b	b	b	e
5	a	a	b	a	c
6	a	c	b	b	b
7	a	d	c	e	c
8	a	c	e	e	d
9	a	b	b	b	e
10	a	e	d	b	e
11	a	c	d	a	c
12	a	e	e	c	a
13	a	c	a	b	d
14	a	c	b	b	d