Instructions:

1. Calculators and Mobiles are not allowed.

2. Write neatly and legibly. You may lose points for messy work.

3. Show all your work. No points for answers without justification.

4. Make sure that you have 6 pages of problems (Total of 10 Problems)
1. Consider the lines

\[ L_1 : x = -4t, \ y = 1, \ z = 4 + t, \ -\infty < t < \infty, \]
\[ L_2 : x = 2s - 4, \ y = s + 1, \ z = 5 - s, \ -\infty < s < \infty. \]

a) (6-points) Find, if it exists, the point at the intersection of \( L_1 \) and \( L_2 \).

\[
\begin{align*}
-4t &= 2s - 4 \\
1 &= s + 1 \\
4t &= 5 - s
\end{align*}
\]

Using the first two equations in this system, we find that \( t = 1 \) and \( s = 0 \). \( 2 \) pts.

We verify that these values satisfy the 3rd eqn.

\[ 4t = 5 - s. \] \( 1 \) pt.

Then the lines \( L_1 \) and \( L_2 \) intersect.

The point of intersection is \((-4, 1, 5)\) which can be found using \( L_1 \) and \( t = 1 \) or \( L_2 \) and \( s = 0 \). \( 2 \) pts.

b) (7-points) Find the symmetric equations of the line passing through the point \((1, 2, 3)\) and perpendicular to \( L_1 \) and \( L_2 \).

The direction vector of \( L_1 \) is \( d_1 = \langle -4, 0, 1 \rangle \) \( 1 \) pt.

The direction vector of \( L_2 \) is \( d_2 = \langle 2, 1, -1 \rangle \) \( 1 \) pt.

The direction of the line perpendicular to both \( L_1 \) and \( L_2 \) is

\[
\mathbf{d_1} \times \mathbf{d_2} = \begin{vmatrix}
1 & 3 & 1 \\
-4 & 0 & 1 \\
2 & 1 & -1
\end{vmatrix} = \langle -1, -2, -4 \rangle \] \( 2 \) pts.

The symmetric equations are:

\[
\frac{x-1}{-1} = \frac{y-2}{-2} = \frac{z-3}{-4} \] \( 3 \) pts.

c) (5-points) Find an equation of the plane containing the lines \( L_1 \) and \( L_2 \).

\[ \mathbf{d_1} \times \mathbf{d_2}, \ as \ in \ part \ (b), \ is \ normal \ to \ the \ plane \]

containing the lines \( L_1 \) and \( L_2 \). \( 1 \) pt.

Using the point of the intersection of \( L_1 \) and \( L_2 \) from part (a), an equation of the plane is \( 1 \) pt.
2. (7-points) Identify the quadric surface \( x^2 + 4y^2 - 2x + 16y - 2z + 18 = 0 \) and its traces on the planes \( z = k \) for \( k \) any real number.

\[
x^2 - 2x + 1 + 4(y^2 + 4y + 4) - 2z + 18 = 1 + 16
\]

\[
(x - 1)^2 + 4(y + 2)^2 = 2z - 1 = \frac{(x-1)^2}{2} + 2(y+2)^2 = z - \frac{1}{2}
\]

this is an elliptical paraboloid \( 2 \text{pts} \)

the equations of the traces are \( \frac{(x-1)^2}{2} + 2(y+2)^2 = k - \frac{1}{2}, k > \frac{1}{2} \)

\[1 \text{ pt} \]

if \( k = \frac{1}{2} \), then \( \frac{(x-1)^2}{2} + 2(y+2)^2 = 0 \). So the trace is the point \( (1, -2, \frac{1}{2}) \). \( 1 \text{ pt} \)

if \( k > \frac{1}{2} \), then the traces are \( \text{ellipses} \). \( 1 \text{ pt} \)

3. Let \( f(x, y) = \frac{1}{\ln(e^2 - x^2 - 4y^2)} \).

(a) (8-points) Find and sketch the domain of \( f \).

The domain of \( f \) is all \((x, y)\) in \( \mathbb{R}^2 \) so that

\[
e^2 - x^2 - 4y^2 > 0 \quad \text{and} \quad e^2 - x^2 - 4y^2 \neq 1
\]

\(2 \text{ pts} \)

\[
\text{Domain} = \{(x, y) \in \mathbb{R}^2 \mid e^2 > x^2 + 4y^2 \quad \text{and} \quad x^2 + 4y^2 \neq e^2 - 1\}
\]

\(2 \text{ pts} \)

\[
\begin{align*}
x^2 + 4y^2 &= e^2 - 1 & \text{x region} & 1 \text{ pt} \\
x^2 + 4y^2 &= e^2 & \text{* including ellipse for each ellipse} \\
\end{align*}
\]

(b) (4-points) Find and identify the level curve \( f(x, y) = \frac{1}{2} \).

\[
\frac{1}{\ln(e^2 - x^2 - 4y^2)} = \frac{1}{2} \Rightarrow \ln(e^2 - x^2 - 4y^2) = 2 \Rightarrow e^2 - x^2 - 4y^2 = e^2
\]

\(1 \text{ pt} \)
4. (6-points) Find the limit or show that it does not exist.

\[ \lim_{(x,y) \to (0,0)} \frac{x^3 + y^3}{\tan^{-1}(x^2 + y^2)} \]

Using polar coordinates: \( x = r \cos \theta, \quad y = r \sin \theta \).

\[ \lim_{(x,y) \to (0,0)} \frac{x^3 + y^3}{\tan^{-1}(x^2 + y^2)} = \lim_{r \to 0} \frac{r^3 (\cos^3 \theta + \sin^3 \theta)}{\tan^{-1}(r^2)} \]

\[ = \lim_{r \to 0} \frac{3r^3 (\cos^3 \theta + \sin^3 \theta)}{2r} \]

\[ = \lim_{r \to 0} \frac{3r (1 + r^4) (\cos^3 \theta + \sin^3 \theta)}{2} \]

\[ = 0 \] \hspace{2cm} 2 pts

5. (8-points) Determine whether \( f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases} \) is continuous at \((0,0)\) or not. Give reasons for your answer.

1 pt \hspace{1cm} f \text{ is continuous at } (0,0) \text{ whenever } \lim_{(x,y) \to (0,0)} f(x,y) = f(0,0).

2 pts \hspace{1cm} \text{along path } x=0, \lim_{(x,y) \to (0,0)} f(x,y) = \lim_{y \to 0} \frac{0}{y} = 0 \quad y \to 0

2 pts \hspace{1cm} \text{along path } y=x^2, \lim_{(x,y) \to (0,0)} f(x,y) = \lim_{x \to 0} \frac{x^5}{x^4 + x^4} = \frac{1}{2}

2 pts \hspace{1cm} \text{Then } \lim_{(x,y) \to (0,0)} f(x,y) \text{ DNE.}

Not so \hspace{1cm} f \text{ is not continuous at } (0,0).
6. (8-points) Let \( f(x, y, z) = xe^{xy} \sin^2 z \). Find \( f_{xyz}(-1, 0, \pi/4) \).

\[
\begin{align*}
    f_x(x, y, z) &= xe^{xy} 2 \sin z \cos z = xe^{xy} \sin (2z) & \text{2pts} \\
    f_y(x, y, z) &= x^2 e^{xy} \sin (2z) & \text{2pts} \\
    f_{yz}(x, y, z) &= 2xe^{xy} \sin(2z) + x^2y e^{xy} \sin(2z) & \text{2pts} \\
    f_{xy}(x, y, z) &= -2 \cdot e^x \sin \left( \frac{\pi}{2} \right) + (-1)^2 \cdot 0 \cdot e^x \sin \left( \frac{\pi}{2} \right) = -2 & \text{2pts}
\end{align*}
\]

7. (10-points) Let \( u = \sqrt{x^3 - y^3} \) where \( x(r, s, t) = r(s + t) \) and \( y(r, s, t) = \frac{r - s}{t} \). Use the chain rule to calculate \( \frac{\partial u}{\partial s} \) at \( (r, s, t) = (3, -1, 2) \).

\[
\begin{align*}
    x(3, -1, 2) &= 3 & 1pt \\
    y(3, -1, 2) &= \frac{3 - (-1)}{2} - 2 & 1pt \\
    \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} & 1pt \\
    \frac{\partial u}{\partial x}(3, 2) &= \frac{1}{2 \sqrt{3x^3 - y^3}} \cdot 3x^2 \bigg|_{(3, 2)} = \frac{27}{2 \sqrt{19}} & 2pts \\
    \frac{\partial u}{\partial y}(3, 2) &= \frac{1}{2 \sqrt{3x^3 - y^3}} \cdot 3y^2 \bigg|_{(3, 2)} = \frac{-12}{2 \sqrt{19}} & 2pts \\
    \frac{\partial x}{\partial s}(3, -1, 2) &= \Gamma \bigg|_{(3, -1, 2)} = 3 & 1pt
\end{align*}
\]
8. Consider the surface given by the equation \( \sin(xz) - 4 \cos(yz) = 0 \) and the point \( P(\pi, -\pi/2, 1) \) on this surface.

(a) (9-points) Find the parametric equations of the normal line to the above surface at the point \( P \).

Consider the given surface as the level surface \( f(x,y,z) = 0 \)
where \( f(x,y,z) = \sin(xz) - 4 \cos(yz) \)

The direction of the normal line to this surface at \( P \) is
\[
\nabla f(P) = \begin{pmatrix} f_x(P) \\ f_y(P) \\ f_z(P) \end{pmatrix}
\]
\[
f_x(x,y,z) = z \cos(xz) \quad \rightarrow \quad f_x(\pi, -\pi/2, 1) = -1 \quad 2 \text{ pts}
\]
\[
f_y(x,y,z) = 4z \sin(yz) \quad \rightarrow \quad f_y(\pi, -\pi/2, 1) = -4 \quad 2 \text{ pts}
\]
\[
f_z(x,y,z) = x \cos(xz) + 4y \sin(yz) \quad \rightarrow \quad f_z(\pi, -\pi/2, 1) = \pi \quad 2 \text{ pts}
\]
Then \( \nabla f(P) = \begin{pmatrix} -1 \\ -4 \\ \pi \end{pmatrix} \)

Parametric equations of the normal line at \( P \) are:
\[
x = \pi - t, \quad y = \frac{-\pi}{2} + 4t, \quad z = 1 - \pi t \quad 3 \text{ pts}
\]

(b) (4-points) Find an equation of the tangent plane to the above surface at the point \( P \).

\( \nabla f(P) = \begin{pmatrix} -1 \\ -4 \\ \pi \end{pmatrix} \) is normal to the tangent plane at \( P \). 1 pt

An equation of this tangent plane is
\[
-1 \cdot (x - \pi) - 4 \cdot \left( y - \frac{-\pi}{2} \right) + \pi (z - 1) = 0
\]
\[
\text{or}
1 + 4y - \pi z = 2\pi
\]
3 pts
9. (9-points) Find the maximum rate of change of \( f(x, y, z) = e^{x^2 - y^2} \) at the point \((-2, 1, -1)\) and the unit vector along which it occurs.

\[
\nabla f(x, y, z) = \langle 2e^{x^2 - y^2}, -2y e^{x^2 - y^2}, x e^{x^2 - y^2} \rangle
\]

\[
\nabla f(-2, 1, -1) = \langle -2, -2e, -2e \rangle
\]

the maximum rate of change is \( |\nabla f(-2, 1, -1)| = 2e \)

the unit vector along which it occurs is \( \frac{-\nabla f(-2, 1, -1)}{|\nabla f(-2, 1, -1)|} = \frac{1}{3} \langle 2, 2, 2 \rangle \)

10. (9-points) Find the linearization of \( f(x, y) = x^2 y - \sqrt{xy} + 1 \) at \((-1, -4)\) and use it to estimate the value of \( f(-1.05, -3.96) \).

\[
L(x, y) = -5 + 9(x + 1) + \frac{5}{2}(y + 4)
\]

\[
L(-1.05, -3.96) \approx L(-1.05, -3.96) = -5 + 9(-0.05) + \frac{5}{2}(0.04)
\]

\[
= -5 - 0.45 + 0.05
\]

\[
= -5.4
\]