Exercise 1. Use Lagrange’s theorem to prove the Fermat’s little theorem: If $p$ is a prime number and $a$ is an integer such that $p$ does not divide $a$, then

$$a^{p-1} \equiv 1 \pmod{p}.$$
Exercise 2. Show that $\mathbb{Q}/\mathbb{Z}$ is an infinite group where every element is of finite order.
Exercise 3. Let \( p \) be a prime number. Determine the number of elements of order \( p \) in \( \mathbb{Z}_p^2 \oplus \mathbb{Z}_p^2 \).
Exercise 4. Is a subgroup of a direct product of groups $G_1 \oplus G_2$ a direct product of subgroups $H_1 \oplus H_2$, where $H_i = -G_i$?
Exercise 5. Determine the number of cyclic subgroups of order 6 in $\mathbb{Z}_{36} \oplus \mathbb{Z}_9$. 
Exercise 6. Consider the group \((\mathbb{R}^*, \times)\)

(1) Find all subgroups \(H\) of \(\mathbb{R}^*\) with index 2.

(2) Show that \(\mathbb{R}^*/H \cong \mathbb{Z}_2\).

(3) Let \(K = \{2^n : n \in \mathbb{Z}\}\). Show that \(K\) is a cyclic subgroup of \(\mathbb{R}^*\). Is \(\mathbb{R}^*/K\) cyclic?
Exercise 7. A group $G$ is called metacyclic if it contains a normal cyclic subgroup $H$ such that $G/H$ is cyclic.

(1) Show that every cyclic group is metacyclic.

(2) Show that $S_3$ is a metacyclic noncyclic group.

(3) Show that in a metacyclic group, every subgroup is metacyclic.