Exercise 1. Determine (up to an isomorphism) all Abelian groups of order 56.
Exercise 2. Let $G$ be an Abelian group,

$$H = \{ x^2 \mid x \in G \} \text{ and } K = \{ x \in G \mid x^2 = e \}.$$ 

(1) Show that $H, K$ are subgroups of $G$.

(2) Show that $G/K \simeq H$. 
Exercise 3. Let $G_1, G_2$ be two groups and $H_1, H_2$ be two normal subgroups of $G_1, G_2$ respectively.

(1) Show that $H_1 \oplus H_2$ is a normal subgroup of $G_1 \oplus G_2$.

(2) Show that $(G_1 \oplus G_2)/(H_1 \oplus H_2) \simeq (G_1/H_1) \oplus (G_2/H_2)$. 
Exercise 4. Let $m, n$ be positive integers and $f : \mathbb{Z}_n \rightarrow \mathbb{Z}_m$ be a morphism of groups.

(1) Show that there exists $0 \leq a \leq m - 1$ such that

$$na \equiv 0 \pmod{m} \text{ and } f(x + n\mathbb{Z}) = ax + m\mathbb{Z},$$

for all $x \in \mathbb{Z}$

(2) Show that the number of morphisms of groups from $\mathbb{Z}_n$ to $\mathbb{Z}_m$ is $d = \gcd(m, n)$. 
Exercise 5. Let $R$ be an integral domain, and

$$N : R - \{0\} \rightarrow \mathbb{N} = \{0, 1, 2, \ldots\}$$

be a function satisfying $N(ab) = N(a)N(b)$, for all $a, b \in R - \{0\}$. Assume the set of all invertible elements of $R$ is $U(R) = \{x \in R - \{0\} | N(x) = 1\}$.

1. Show that if $x \in R - \{0\}$ and $N(x)$ is prime, then $x$ is irreducible.

2. Let $x \in R - \{0\}$ and $y$ be a non-invertible divisor of $x$ which is not associated with $x$ (i.e., there is no $\varepsilon \in U(R)$ such that $y = \varepsilon x$). Suppose that $N(x) = p^2$, for some prime number $p$; show that $p = N(y)$. 
Exercise 6. Write $z = -33 + 9i$ as a product of irreducibles in $\mathbb{Z}[i]$. 
Exercise 7. Let $R$ be an integral domain.

(1) Show that if $R$ is a PID, then every nonzero prime ideal of $R$ is maximal.

(2) Show that $R[X]/(X.R[X]) \simeq R$. Deduce that $X.R[X]$ is a prime ideal of $R[X]$.

(3) Show that $R[X]$ is a PID if and only if $R$ is a field.
Exercise 8. Let $R := \mathbb{Z}[i\sqrt{3}] = \{a + ib\sqrt{3} \mid a, b \in \mathbb{Z}\}$.

(1) Show that $R$ is a subring of $\mathbb{C}$ with quotient field
\[ \mathbb{Q}(i\sqrt{3}) := \{a + ib\sqrt{3} \mid a, b \in \mathbb{Q}\}. \]

(2) Show that $R \simeq \mathbb{Z}[X]/(X^2 + 3)\mathbb{Z}[X]$.

(3) Is $(X^2 + 3)\mathbb{Z}[X]$ a maximal ideal of $\mathbb{Z}[X]$?

(4) Find the set of all invertible elements of $R$.

(5) Show that $2, 1 + i\sqrt{3}, 1 - i\sqrt{3}$ are irreducible elements of $R$; and deduce that $R$ is not a UFD.