Exercise 1

Let

\[ f(x, y) = \begin{cases} \frac{xyz}{(x^2 + y^2 + z^2)^\alpha} & \text{if } (x, y, z) \neq (0, 0, 0) \\ 0 & \text{if } (x, y, y) = (0, 0, 0) \end{cases} \]

where \( \alpha \in \mathbb{R} \) is a constant. Show that \( f \) is differentiable at \((0, 0, 0)\) if and only if \( \alpha < 1 \).
Exercise 2

Let \( u = xyf\left(\frac{x+y}{xy}\right) \), where \( f : \mathbb{R} \rightarrow \mathbb{R} \) is differentiable. Show that \( u \) satisfies the equation

\[ x^2 \frac{\partial u}{\partial x} - y^2 \frac{\partial u}{\partial y} = g(x, y)u \]

and find \( g(x, y) \).
Exercise 3
For $x \in \mathbb{R}^n \setminus \{0\}$, let $f(x) = g(||x||)$ where $g$ is $C^2$ on $(0, \infty)$. Show that

$$\frac{\partial^2 f}{\partial^2 x_1} + \ldots + \frac{\partial^2 f}{\partial^2 x_n} = g''(r) + \frac{n-1}{r} g'(r)$$

where $r = ||x||$.
Deduce that $f(x) = ||x||^{2-n}, \ n \geq 3$, is harmonic.
Exercise 4
Let $f(x, y) = e^{xy} \sin(x + y)$. Find the Taylor polynomial of $f$ of order 3 about the point $(0, 0)$. 
Exercise 5
Find the critical points of \( f(x, y, z) = x^3 - y^3 + z^2 - 3x + 9y \) and determine their nature.
Exercise 6
Find the maximum of $x^2y^2z^2$ on the sphere $x^2 + y^2 + z^2 = R^2$. 
Exercise 7
Let \( R \subset \mathbb{R}^n \) be a rectangle and \( f : R \to \mathbb{R} \) be a bounded function. Define the following

1. \( f \) is Riemann integrable
2. \( f \) is Darboux integrable.

Give an example of a function \( f \) on \([0,1] \times [0,1]\) which is not Riemann integrable.
Exercise 8

Let $f : \Omega \to \mathbb{R}$ be an integrable function over $\Omega$ a bounded simple subset of $\mathbb{R}^n$

Show that

1. If $f$ is almost everywhere zero, then $\int_{\Omega} f = 0$.
2. If $f \geq 0$ and $\int_{\Omega} f = 0$, then $f$ is almost everywhere zero.