Show all your work. No credits for answers without justification.
Write neatly and eligibly. You may lose points for messy work.

Make sure that you have 9 pages with 7 questions.

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Problem 1 [16 marks]

(A) [8 marks] Define each of the following

(a) A 1-tough graph:

(b) The line graph of a plane graph $G$ of order $n$ and size $m$:

(c) Perfect Matching:

(d) Edge cover:

(B) [8 marks] State each of the following; define the terminology you use in the theorems:

a) The Matrix-Tree Theorem.

b) Kutatowski’s Theorem for planar graphs.

c) Menger’s Theorem.

d) Orthogonality relation between matrices of graphs.
Problem 2 [14 marks]

(A) [8 marks] Use Havel-Hakimi Theorem to determine whether the sequence is graphical or not. If yes draw the corresponding graph. (7,4,3,3,2,2,2,1)

(B) [6 marks] If exists, find a maximum matching and a minimum vertex cover in the following graph.
Problem 3 [18 marks]

(A) [6 marks] Determine the connectivity and the edge-connectivity of the graph from the picture.

(B) [12 marks] For the graphs $G_1$ and $G_2$ from the picture, prove non-planarity or provide a planar embedding.
Problem 4 [18 marks]

(A) [6 marks]: Determine whether the given graph is Hamiltonian. If it is, find a Hamiltonian cycle. If it is not, prove it is not.

(B) [12 marks] Consider the network with source $s$ and sink (terminal) $t$, and with the given capacity. Find a maximum flow. Justify your answer.
Problem 5 [22 marks]

(A) [12 marks] Let \( G \) be a graph of order \( n > 4 \) such that \( d(v) \geq \frac{n-1}{2} \) for all vertices \( v \) of \( G \). Prove that:

(a) \( G \) is connected.

(b) \( G \) contains a cycle.

(c) \( diam(G) \leq 2 \).

(d) \( G \) contains a Hamiltonian path.

(B) [10 marks] Answer each of the following. Sketch The graph if possible.

(a) The crossing number of \( K_{1,2,3} \) is equal to: ____

(b) A maximal outer planar graph of order \( n \) must have size: __

(c) Find all connected graphs \( G \) where \( G \cong L(G) \).

(d) Find a connected plane graph \( G \) which is isomorphic to its dual \( G^* \)

(e) If \( P_n \) is a path of order \( n \) (\( n \geq 2 \)), then \( |Aut(P_n)| = ____ \)
Problem 6 [20 marks]
(A) [10 marks] In a village there are three schools with \( n \) students in each of them. Every student from any of the schools is on speaking terms with at least \( n + 1 \) students from the other two schools. Show that we can find three students, no two from the same school, who are on speaking terms with each other.

(B) [10 marks] Consider the digraph \( G \) with the spanning tree \( T = \{e_1, e_3, e_4\} \).

(a) Find the fundamental cutset matrix \( Q_f \) with respect to \( T \).

(b) Find the fundamental circuit matrix \( B_f \) with respect to \( T \).

(c) Arrange the columns of both \( Q_f \) and \( B_f \) in the same edge order and calculate \( B_f Q_f^t \).
Problem 7 [32 marks] For each of the following statements decide if it is true or false. Give a succinct explanation.

1. Every 3-regular graph has a perfect matching.

2. There exists a 6-connected planar graph.

3. Every connected graph of order $n$ and size $n - 1$ is a tree.

4. The complete graph $K_{2n+1}$ can be factored into Hamiltonian paths.
5. If a graph $G$ has exactly two vertices $u$ and $v$ of odd degree, then $G$ has a $u - v$ path.

6. If $v$ is a cut vertex of a connected graph $G$, then $v$ is a cut vertex of the complement $\overline{G}$.

7. Any cutest and any cycle of a graph have an even number of edges in common.

8. Every tournament contains a Hamiltonian path.