(1) (a) What does it mean to say that a function \( f : \mathbb{R} \to \mathbb{R} \) is measurable?

**Solution:** An extended real-valued function \( f \) defined on \( \mathbb{R} \) is said to be Lebesgue measurable, or simply measurable, provided it satisfies one of the following four equivalent conditions:

(i) For each real number \( c \), the set \( \{ x : f(x) > c \} \) is measurable.

(ii) For each real number \( c \), the set \( \{ x : f(x) \geq c \} \) is measurable.

(iii) For each real number \( c \), the set \( \{ x : f(x) < c \} \) is measurable.

(iv) For each real number \( c \), the set \( \{ x : f(x) \leq c \} \) is measurable.

(b) Prove that if \( f : \mathbb{R} \to \mathbb{R} \) is increasing (i.e. \( f(x) \leq f(y) \) whenever \( x \leq y \)) then it is measurable.

**Solution:**

(1) If \( f \) is increasing, the set \( \{ x \in \mathbb{R} : f(x) > a \} \) is an interval for all \( a \), hence measurable. Therefore, by the definition (see (a) above), the function \( f \) is measurable.

(2) Let \( D \) be the set of discontinuities of \( f \). Then \( D \) is countable, hence of measure zero. The restriction \( f|_D \) is measurable on \( D \) because every subset of \( D \) is measurable, and the restriction \( f|_{\mathbb{R} \sim D} \) is measurable on \( \mathbb{R} \sim D \) because it is continuous. Therefore, \( f \) is measurable (see Proposition 5 - Section 3,1).

(c) Suppose that \( f : [0, 1] \to \mathbb{R} \) is measurable and that there is \( \delta > 0 \) such that, for each \( n \in \mathbb{N} \), \( m\{x : |f(x)| \leq 1/n\} \geq \delta \).

(i) Explain why \( \{ x : |f(x)| \leq 1/n \} \) is measurable.

(ii) Explain why there is at least one \( s \in [0, 1] \) such that \( f(s) = 0 \).

**Solution:**

(i) Since \( f \) is measurable, \(|f|\) is measurable as its the composition of continuous function \( g(x) = |x| \) with a measurable function \( f \) and \( \{x : |f(x)| \leq 1/n\} = |f|^{-1}[0, 1/n] \). Since \([0, 1/n] \) is a Borel set, \(|f|^{-1}[0, 1/n] \) is measurable.

(ii) Let \( E_n = \{ x : |f(x)| \leq 1/n \} \). Then \( E_n \supset E_{n+1} \), \( \cap E_n = \{ x : f(x) = 0 \} \),
$E_1 \subset [0, 1]$. So $m\{x : f(x) = 0\} = \lim_{n \to \infty} m(E_n) \geq \delta$ (Excision property of $m$). Then $\{x : f(x) = 0\} \neq \emptyset$ (Since $m(\emptyset) = 0$). So $\exists s$ so that $f(s) = 0$.

(2) (a) For a measurable subset $E \subseteq \mathbb{R}$, and simple function $\varphi : \mathbb{R} \to \mathbb{R}$, how is the (Lebesgue) integral $\int_E \varphi \, dm$ defined?

**Solution:** For a simple function $\varphi$ defined on a set of finite measure $E$, we define the integral of $\varphi$ over $E$ by

$$\int_E \varphi = \sum_{i=1}^{n} a_i m(E_i),$$

where $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ and $E_i = \{x \in E : \varphi(x) = a_i\}$.

(b) State Fatou’s Lemma for a sequence of measurable functions.

**Solution:** Let $\{f_n\}$ be a sequence of measurable functions on $E$. If $\{f_n\} \to f$ pointwise a.e. on $E$, then

$$\int_E f = \int_E \lim f_n \leq \lim inf \int_E f_n.$$

(c) State the Monotone Convergence Theorem.

**Solution:** Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on $E$. If $\{f_n\} \to f$ pointwise a.e. on $E$, then

$$\lim_{n \to \infty} \left( \int_E f_n \right) = \int_E \left( \lim_{n \to \infty} f_n \right) = \int_E f.$$

(d) Prove that Fatou’s Lemma implies the Monotone Convergence Theorem.

**Solution:** According to Fatou’s Lemma,

$$\int_E f \leq \lim inf \int_E f_n.$$

Also, notice that if $f$ is a nonnegative measurable function on $E$ and $E_0$ is a subset of $E$ of measure zero, then

$$\int_E f = \int_{E \sim E_0} f \quad (\ast).$$

However, for each $n, f_n \leq f$ a.e. on $E$ (note that $f$ is measurable), and by the monotonicity of integration for nonnegative measurable functions and $(\ast)$, $\int_E f_n \leq \int_E f$. Therefore,

$$\lim sup \int_E f_n \leq \int_E f.$$

Hence $\int_E f = \lim_{n \to \infty} \int_E f_n$. 
(3) Identify which of the following statements is true and which is false. If a statement is true, give reason. If a statement is false, provide a counterexample.

(a) If \( f \) is a bounded real-valued function on \([0, 1]\) which is Lebesgue integrable then \( f \) is Riemann integrable.

**Solution:** False. Consider the Dirichlet function \( f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0, 1]; \\ 0 & x \in [0, 1] \sim \mathbb{Q}. \end{cases} \)

(b) Suppose that \((E_n)\) is a sequence of pairwise disjoint measurable subsets of \([0, 1]\). Then \( \lim_{n \to \infty} m(E_n) = 0. \)

**Solution:** True, indeed since \( E_n \subset [0, 1] \ \forall n \in \mathbb{N} \), then \( \bigcup_{n=1}^{\infty} E_n \subset [0, 1] \). By monotonicity of the measure \( m(\bigcup_{n=1}^{\infty} E_n) \leq m([0, 1]) = 1 \). Hence \( \sum_{n=1}^{\infty} m(E_n) \leq 1 \) since \( m \) is countably additive. Thus \( \lim_{n \to \infty} m(E_n) = 0. \)

(c) If \( f(x) = \int_{\mathbb{R}} \frac{(\sin t)^2}{t^2 + x^2} \, dt \), then \( \lim_{x \to \infty} f(x) = 0. \)

**Solution:** \( \frac{(\sin t)^2}{t^2 + x^2} \leq \frac{1}{t^2 + x^2} \) for all \( t \). Hence by monotonicity of Riemann integrable functions \( \int_{\mathbb{R}} \frac{(\sin t)^2}{t^2 + x^2} \, dt \leq \int_{\mathbb{R}} \frac{1}{t^2} \, dt = \frac{1}{x^2} \int_{\mathbb{R}} \frac{1}{(\frac{t}{x})^2 + 1} \, dt = \frac{1}{x^2} [\tan^{-1} \frac{t}{x}]_{-\infty}^{\infty} = \frac{\pi}{x^2} < \infty. \) Since \( f(x) \) is positive and limit of \( \frac{\pi}{x^2} \) goes to zero as \( x \) goes to \( \infty \), then \( \lim_{x \to \infty} f(x) = 0. \)

(4) (a) State Egoroff’s Theorem.

Assume \( E \) has finite measure. Let \( \{f_n\} \), be a sequence of measurable functions on \( E \) that converges pointwise on \( E \) to the real-valued function \( f \). Then for each \( \epsilon > 0 \), there is a closed set \( F \) contained in \( E \) such that \( \{f_n\} \to f \) uniformly on \( F \) and \( m(E \sim F) < \epsilon. \)

(b) Let \( f \) be a real-valued measurable function defined on \([0, 1]\). Prove that for each \( \epsilon > 0 \) there is a measurable set \( E_\epsilon \subset [0, 1] \) so that \( m([0, 1] \sim E_\epsilon) < \epsilon \) and so that \( f \) is bounded on \( E_\epsilon. \)

**Solution** (1) (Using Egoroff’s Theorem) Let \( f_n = f \chi_{\{|f| \leq n\}}. \) Since \( f \) is measurable, \( \{x : |f(x)| \leq n\} = f^{-1}[-n, n] \) is measurable. So, \( \chi_{\{|f| \leq n\}} \) is a measurable function and hence \( f \chi_{\{|f| \leq n\}} \), the product of two measurable functions is measurable. If \( |f(x)| < N \) then \( f_n(x) = f(x) \) for all \( n \geq N. \) So, \( \lim_{n \to \infty} f_n(x) = f(x) \ \forall x. \) Then by Egoroff’s Theorem \( \forall \epsilon > 0 \ \exists E_\epsilon \subset [0, 1] \) such that \( m([0, 1] \sim E_\epsilon) < \epsilon \) and \( f_n \to f \) uniformly on \( E_\epsilon. \) Since \( f_n \to f \) uniformly, in particular, \( \exists N \) such that \( |f_n(x) - f(x)| < 1 \) for all \( n \geq N \) and \( x \in E_\epsilon. \) Thus \( |f(x)| < 1 + |f_n(x)| \leq N + 1 \) on \( E_\epsilon. \)
Or (2) Let $E_n = \{ x : |f(x)| \geq n \}$. Then $E_{n+1} \subset E_n$, $\cap E_n = \emptyset$ and $m(E_1) \leq m[0,1] = 1$. Since $f$ is real valued function, then $\forall \epsilon > 0 \exists N$ such that $m(E_N) < \epsilon$. And $|f(x)| \leq N$ on $[0,1] \sim E_N$.

(5) Suppose that $f$ is integrable on $[0,1]$. Let $p_n(x) = x^n, n \in \mathbb{N}$.

(a) State why, for each $n$, $fp_n$ is measurable and integrable on $[0,1]$.

**Solution:** $p_n$ is continuous on $[0,1]$ and so $p_n$ is measurable. Then $fp_n$, the product of two measurable functions is measurable. Moreover, $|p_n| \leq 1$ on $[0,1]$, so $|fp_n| \leq |f|$ and since $f$ is integrable so is each $fp_n$.

(b) Prove that $\lim_{n \to \infty} \int_{[0,1]} fp_n dm = 0$.

**Solution:** Now $\lim_{n \to \infty} f(x)p_n(x) = 0$ unless $x = 1$ or $|f(x)| = \infty$. So, $\lim_{n \to \infty} f(x)p_n(x) = 0$ a.e. Since $|fp_n| \leq |f|$ we may apply the Dominated Convergence Theorem to get $\lim_{n \to \infty} \int_{[0,1]} fp_n dm = \int_{[0,1]} \lim_{n \to \infty} fp_n dm = 0$.

(6) (a) State the Dominated Convergence Theorem.

**Solution** Let $\{f_n\}$ be a sequence of measurable functions on $E$. Suppose there is a function $g$ that is integrable over $E$ and dominates $\{f_n\}$ on $E$ in the sense that $|f_n| \leq g$ on $E$ for all $n$. If $\{f_n\} \to f$ pointwise a.e. on $E$, then $f$ is integrable over $E$ and $\lim_{n \to \infty} \int_E f_n = \int_E f$.

(b) Use the Dominated Convergence Theorem to find

$$\lim_{n \to \infty} \int_0^\infty f_n dm,$$

where for each $n \geq 1$ the function $f_n : [0, \infty) \to \mathbb{R}$ is defined by

$$f_n(x) = \frac{x \sin \pi nx}{1 + nx^3}.$$

**Solution:** For $n \geq 1$, we have $|f_n(x)| = \frac{|x \sin \pi nx|}{1 + nx^3} \leq \frac{x}{1 + nx^3} \leq \frac{x}{nx^3} = \frac{1}{nx^2} \leq \frac{1}{x^2}$ for all $x \in (0, \infty)$. Also note that the function $\frac{1}{x^2}$ is integrable over $[0, \infty)$ ($\int_0^{\infty} \frac{1}{x^2} = \int_0^{\infty} \frac{1}{x^2} < \infty$).

Thus, by the Dominated Convergence Theorem and the squeezing theorem, we have

$$\lim_{n \to \infty} \int_{(0,\infty)} f_n = \int_{(0,\infty)} 0 = 0.$$

Notice that $\int_E f = \int_{E \sim E_0} f$ if $m(E_0) = 0$. 4
(7) (a) State Beppo Levis Theorem.

**Solution** Let \( \{f_n\} \) be an increasing sequence of nonnegative measurable functions on \( E \). If the sequence of integrals \( \{\int_E f_n\} \) is bounded, then \( \{f_n\} \) converges pointwise on \( E \) to a measurable function \( f \) that is finite a.e on \( E \) and

\[
\lim_{n \to \infty} \int_E f_n = \int_E f < \infty.
\]

(b) Use Beppo Levis Theorem, and the fact that \( \sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6} \), to prove that

\[
\int_0^\infty \frac{x}{e^x - 1} \, dx = \frac{\pi^2}{6}.
\]

**Solution:** First notice that \( \frac{x}{e^x - 1} = \frac{xe^{-x}}{1 - e^{-x}} \). Now, using the Geometric serious \( 1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1-a} \), if \( |a| < 1 \), we have \( \frac{1}{1-e^{-x}} = \sum_{n=0}^{\infty} e^{-nx} \) for \( x > 0 \). So,

\[
\frac{x}{e^x - 1} = \frac{xe^{-x}}{1 - e^{-x}} = \sum_{n=0}^{\infty} xe^{-(n+1)x}.
\]

Let \( f(x) = \frac{x}{e^x - 1} \) and define the sequence \( (f_n) \) by \( f_n(x) = f(0,n] \) for each \( n \geq 1 \). Notice that \( (f_n) \) is an increasing sequence of nonnegative measurable functions \( (f_n) \) is the product of two measurable functions \( (0,n] \) is measurable since \( (0,n] \) is measurable and \( f \) is measurable since it is continuous on \( (0, \infty) \). Moreover, \( f_n \to f \) a.e. on \( [0, \infty) \). Using Beppo Levis Theorem and integration by parts, we have

\[
\lim_{n \to \infty} \int_0^\infty f_n \, dx = \int_0^\infty f \, dx = \int_0^\infty \frac{x}{e^x - 1} \, dx = \int_0^\infty \sum_{n=0}^{\infty} xe^{-(n+1)x} \, dx = \sum_{n=0}^{\infty} \int_0^\infty xe^{-(n+1)x} \, dx = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]

Dr. M. R. Alfuraidan