Department of Mathematics and Statistics, KFUPM

Final Exam for Math 571, 19 May 2017

Note: In all questions, $f(x,y)$ is smooth and is assumed to satisfy a Lipschitz condition in the second variable. All linear multi-step methods are starting with consistent initial data.

Problem 1.

a) Write down the general form of a linear multistep method for the numerical solution of the IVP: $y' = f(x,y)$ with $y(x_0) = y_0$.

b) What is the truncation error of such a linear multistep method?

Problem 2. Assume that the BVP:

$$y'' = f(x,y), \quad x \in (0,1), \quad y(0) = y(1) = 0,$$

has a unique (smooth) solution. Divide the interval $[0,1]$ uniformly into N subinterval each is of length h and let $x_n = nh$. Assume further that the finite difference solution $y^n \approx y_n := y(x_n)$ defined below is unique.

$$y^{n+1} - 2y^n + y^{n-1} = h^2 f^n, \quad y^0 = y^N = 0, \quad \text{with } f^n = f(x_n, y^n).$$

a) Show that the truncation error $|T_h^n| \leq \frac{h^2}{6}|y^{(4)}(\xi_n)|$ for some ξ_n in the interval (x_{n-1}, x_{n+1}).

b) Show that the global error $e_n = y(x_n) - y^n$ satisfies: $e^{n+1} - 2e^n + e^{n-1} = h^2(T_h^n + q_n e_n)$, where $q_n = \frac{\partial}{\partial y} f(x_n, \eta_n)$ for some η_n between $y(x_n)$ and y^n.

Problem 3. Let γ be a positive real number. Consider the linear two-step method

$$y^{n+2} - \gamma y^n = \frac{h}{3}(f^{n+2} + 4f^{n+1} + f^n).$$

a) Find the values of γ such that the method is zero-stable.

b) Investigate the absolute stability of the above method using Schurs criterion.

c) Determine the order of accuracy when $\gamma = 1$.

Problem 4. If the second characteristic polynomial of a linear multistep method is $\sigma(z) = z^2$, find the first characteristic (quadratic) polynomial $\rho(z)$ such that the method is second order accurate. Is this method convergent?

Problem 5. Show that the linear three-step method

$$11y^{n+3} + 27y^{n+2} - 27y^{n+1} - 11y^n = 3h[f^{n+3} + 9f^{n+2} + 9f^{n+1} + f^n],$$

is consistent and at least first order accurate but not convergent.

Problem 6. A predictor P and a corrector C are defined by their characteristic polynomials:

$$P:\ \rho^*(z) = z^2 - z, \quad \sigma^*(z) = \frac{1}{2}(3z-1),$$

and

$$C:\ \rho(z) = z^2 - 1, \quad \sigma(z) = \frac{1}{3}(z^2+4z+1).$$

Problem 7. Use Routh–Hurwitz criterion to find the interval of absolute stability for the two-step method:

$$y^{n+2} - y^{n+1} = \frac{h}{2}(3f^{n+1} - f^n).$$

Write down algorithms which use P and C in the $P(EC)^m E$ mode.

Problem 8. Show that the θ method below is A-stable for $\theta \geq 1/2$.

$$y^{n+1} - y^n = h\Big[\theta f^{n+1} + (1-\theta) f^n\Big], \quad \theta \in [0,1].$$