Key

Serial No.:_______ Student Name:______________________ Student Number:_______

Instructor: M. Z. Abu-Sbeih Math 101- Q2 Date: 22-10-2017

SHOW ALL YOUR WORK. NO CREDITS FOR ANSWERES WITHOUT JUSTIFICATIONS

Show all your work. NO credits for answers not supported by work.

(1) (5 points) Find all vertical and horizontal asymptotes of the function $f(x) = \frac{x}{x+1} - \frac{1}{x^2+x}$.

$$f(x) = \frac{x}{x+1} - \frac{1}{x(x+1)} = \frac{x^2-1}{x(x+1)} = \frac{(x-1)(x+1)}{x(x+1)}$$

Vertical asymptotes: Check at $x=0$ and $x=-1$

$$\lim_{x\to 0^+} \frac{(x-1)\cancel{(x+1)}}{x\cancel{(x+1)}} = -\infty \Rightarrow \boxed{x=0 \text{ is VA}}$$

$$\lim_{x\to -1} \frac{(x-1)\cancel{(x+1)}}{x\cancel{(x+1)}} = \frac{-2}{-1} = 2 \Rightarrow x=-1 \text{ is NOT VA}$$

Horizontal Asym. $\lim_{x\to\mp\infty} \frac{x^2-1}{x^2+x} = 1 \Rightarrow \boxed{y=1 \text{ is HA}}$

(2) (5 points) If $f'(3)=5$ then $\lim_{x\to 3}\frac{f(x)-f(3)}{x^2-9} =$

$$= \lim_{x\to 3} \frac{f(x)-f(3)}{(x-3)} \cdot \frac{1}{x+3}$$

$$= \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} \cdot \lim_{x\to 3} \frac{1}{x+3}$$

$$= f'(3)\cdot\frac{1}{6} = \frac{5}{6}.$$

(3) (5 points) Use the Intermediate Value Theorem to show that the two functions

$$f(x) = x^2 - 1 \text{ and } g(x) = \sqrt{x+2}$$

Intersect between 0 and 2.

For the two functions to intersect, $f(x)=g(x)$ OR $x^2-1=\sqrt{x+2}$

Let $H(x) = x^2-1-\sqrt{x+2}$

(i) $H(x)$ is continuous on $[0,2]$

(ii) $H(0) = 0-1-\sqrt{2} = -1-\sqrt{2} < 0$

(iii) $H(2) = 2^2-1-\sqrt{2+2} = 3-2 = 1 > 0$

$\therefore$ By the IVT, there is a number c between 0 and 2 such that $H(c) = 0 = c^2-1-\sqrt{c+2}$.

OR $c^2-1=\sqrt{c+2}$.

(4) (5 points) Evaluate the limit: $\lim_{x\to\infty}\left[\sqrt{x^2+4x}-x\right] =$

$$=\lim_{x\to\infty}\left[\sqrt{x^2+4x}-x\right]\cdot\frac{\sqrt{x^2+4x}+x}{\sqrt{x^2+4x}+x} = \lim_{x\to\infty}\frac{x^2+4x-x^2}{\underbrace{\sqrt{x^2+4x}}_{x\sqrt{1+\frac{4}{x}}}+x}$$

$$=\lim_{x\to\infty}\frac{4x}{x\left[\sqrt{1+\frac{4}{x}}+1\right]}$$

$$=\lim_{x\to\infty}\frac{4}{\sqrt{1+\frac{4}{x}}+1} = \frac{4}{1+1} = 2$$

(5) (7 points) Find all points of discontinuity of the function, if any exists, and state the type of discontinuity of each one.

$$f(x)=\begin{cases}\dfrac{1}{x+1} & \text{if } x<-1\\ \sin(x^2-1) & \text{if } -1\le x\le 1\\ \sqrt{x-1} & \text{if } 1<x\end{cases}$$

Note that f is continuous on $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

<u>At $x=-1$:</u>

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}\frac{1}{x+1} = -\infty$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}\sin(x^2-1) = 0$$

$\therefore$ at $x=-1$ we have infinite discontinuity

<u>At $x=1$:</u>

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\sin(x^2-1) = 0$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\sqrt{x-1} = 0$$

$\therefore \lim_{x\to 1} f(x) = 0 = f(1)$. Thus $f(x)$ is continuous at 1.

(6) (5 points) Let $f(x) = \sqrt{2x+1}$. Use the definition of the derivative to find $f'(4)$.

$$f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \to 4} \frac{\sqrt{2x+1} - 3}{x - 4} \qquad \frac{0}{0}$$

$$= \lim_{x \to 4} \frac{\sqrt{2x+1} - 3}{x - 4} \cdot \frac{\sqrt{2x+1} + 3}{\sqrt{2x+1} + 3}$$

$$= \lim_{x \to 4} \frac{2x + 1 - 9}{(x-4)\left[\sqrt{2x+1} + 3\right]}$$

$$= \lim_{x \to 4} \frac{2\cancel{(x-4)}}{\cancel{(x-4)}\left[\sqrt{2x+1} + 3\right]}$$

$$= \lim_{x \to 4} \frac{2}{\sqrt{2x+1} + 3} = \frac{2}{\sqrt{8+1} + 3}$$

$$= \frac{2}{3+3} = \frac{2}{6} = \frac{1}{3}$$

(7) (6 points) Determine if the function $f(x) = \left[\!\left[\frac{x}{1-x}\right]\!\right]$ is continuous at $x = -1$.

$$f(-1) = \left[\!\left[\frac{-1}{1-(-1)}\right]\!\right] = \left[\!\left[-\frac{1}{2}\right]\!\right] = -1$$

<u>Now:</u> $$\lim_{x \to -1^-} \left[\!\left[\frac{x}{1-x}\right]\!\right] = \left[\!\left[\lim_{x \to -1^-} \frac{x}{x-1}\right]\!\right] = \left[\!\left[\frac{-1}{2}\right]\!\right] = -1$$

$$\lim_{x \to -1^+} \left[\!\left[\frac{x}{1-x}\right]\!\right] = \left[\!\left[\lim_{x \to -1^+} \frac{x}{x-1}\right]\!\right] = \left[\!\left[-\frac{1}{2}\right]\!\right] = -1$$

$$\therefore \lim_{x \to -1} f(x) = -1 = f(-1).$$

So $f(x)$ is continuous at $x = -1$

(8) (5 points) Find the rate of change in the surface area of a sphere with respect to the radius when the radius is 5. Here the surface area of a sphere is $S(r) = 4\pi r^2$.

$$\left.\frac{dS}{dr}\right|_{r=5} = S'(5) = \lim_{r\to 5} \frac{S(r)-S(5)}{r-5}$$

$$= \lim_{r\to 5} \frac{4\pi r^2 - 4\pi(25)}{r-5}$$

$$= 4\pi \lim_{r\to 5} \frac{r^2-25}{r-5}$$

$$= 4\pi \lim_{r\to 5} \frac{\cancel{(r-5)}(r+5)}{\cancel{(r-5)}}$$

$$= 4\pi(5+5)$$

$$= 40\pi.$$

(9) (7 points) A particle is moving along a horizontal line with position function

$$s(t) = t - t^2, \qquad \text{for } t \geq 0$$

where $s(t)$ is measured in meters and t in seconds. Find the distance traveled during the first two seconds. Also find the **speed** at = 2 .

$$V(t) = s'(t) = \lim_{h\to 0} \frac{s(t+h)-s(t)}{h} = \lim_{h\to 0} \frac{(t+h)-(t+h)^2-(t-t^2)}{h}$$

$$= \lim_{h\to 0} \frac{t+h-(t^2+2th+h^2)-t+t^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(1-2t+h)}{\cancel{h}} = 1-2t.$$

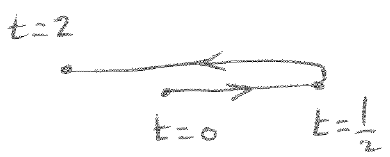
$V(t) = 1-2t = 0 \Rightarrow t = \frac{1}{2}$ Thus: $V(t)$ + | − over 0, $\frac{1}{2}$

The particle reverse direction at $t = \frac{1}{2}$.

Distance traveled $= |S(\tfrac{1}{2}) - S(0)| + |S(2) - S(\tfrac{1}{2})|$

$$= \left|\frac{1}{2} - \frac{1}{4}\right| + \left|(2-4) - \left(\frac{1}{2} - \frac{1}{4}\right)\right|$$

$$= \frac{1}{4} + 2\frac{1}{4} = \frac{5}{2} \text{ m}$$



Speed $|V(2)| = |1-2(2)| = |-3| = 3$ m/s