Problem 1 (10 points) Find the limit if it exists. Write $\infty$ or $(-\infty)$ when appropriate:

a. \[ \lim_{{x \to 0^+}} \sqrt{x} \left(1 + \ln x\right) \]

\[ = \lim_{{x \to 0^+}} \frac{1 + \ln x}{x^{\frac{1}{2}}} \]

\[ = \lim_{{x \to 0^+}} \frac{1}{x^{\frac{1}{2}}} \cdot \frac{1}{x^{-\frac{3}{2}}} \]

\[ = \lim_{{x \to 0^+}} -2 \frac{x}{x^{\frac{3}{2}}} = 0 \]

b. \[ \lim_{{x \to 0^+}} \left(\frac{\ln x}{x}\right)^{\frac{1}{\ln x}} \]

Let \( y = \left(\frac{\ln x}{x}\right)^{\frac{1}{\ln x}} \)

\[ \ln y = \frac{1}{\ln x} \ln \left(\frac{\ln x}{x}\right) \]

\[ \lim_{{x \to 0^+}} \ln y = \lim_{{x \to 0^+}} \frac{\ln (\ln x) - \ln x}{\ln x} \]

\[ = \lim_{{x \to 0^+}} \frac{1 - x}{x \ln x} = -1 \]

\[ \therefore \lim_{{x \to 0^+}} y = e^{-1} = \frac{1}{e} \]
Problem 2 (5 points) Find the absolute maximum and absolute minimum values (if any exist) of the function
\[ f(x) = \sqrt{1-x^2} + \frac{1}{2} \sin^{-1}x \]
on the interval \([0, \frac{\pi}{2}]\).

\[ f'(x) = \frac{-2x}{2\sqrt{1-x^2}} + \frac{1}{2} \frac{1}{\sqrt{1-x^2}} = \frac{-2x+1}{2\sqrt{1-x^2}} \]

Critical nos. \(x = \frac{1}{2}, \ x = \frac{\pi}{2} \).

\[ f(-1) = 0 + \frac{1}{2} \sin^{-1}(-1) = -\frac{1}{2}(\frac{\pi}{2}) = -\frac{\pi}{4} \]

\[ f(\frac{1}{2}) = \sqrt{1-\frac{1}{4}} + \frac{1}{2} \sin^{-1}(\frac{1}{2}) = \frac{\sqrt{3}}{2} + \frac{1}{2} \frac{\pi}{6} = \frac{\sqrt{3}+\pi}{12} \]

Note: \(\frac{\pi}{4} \approx 0.78 \) and \(\frac{\sqrt{3}}{2} \approx 0.866\).

\( \bullet \) Absolute max. value is \(\frac{\sqrt{3}+\pi}{12}\) at \(x = \frac{1}{2}\).

\( \bullet \) Absolute min. value is \(-\frac{\pi}{4}\) at \(x = -1\).

Problem 3 (5 points) show that the equation \(x^3 + 4x - 3 = 0\) has exactly one real root in the interval \([0, 1]\).

Let \(f(x) = x^3 + 4x - 3\).

\(f(x)\) is continuous on \([0, 1]\) and differentiable.

\[ f(0) = -3 < 0 \]
\[ f(1) = 2 > 0 \]

Then there is a number \(b \in (0, 1)\) such that \(f(b) = 0\).

Now \(f' = 3x^2 + 4\).

If there is another root \(c\) such that \(c \in (0, 1)\) and \(f(c) = 0\), then by the MVT there is a number \(d \in (0, 1)\) such that

\[ 3d^2 + 4 = \frac{f(c) - f(b)}{c - b} = \frac{0}{c - b} = 0 \]

i.e. \(3d^2 = -4\) which is impossible.
**Question 4: (31 points)** Consider the function 

\[ y = f(x) = \frac{2x^2 - 3x}{x - 2} \]

with \( f'(x) = \frac{2(x^2 - 4x + 3)}{(x - 2)^2} \) and \( f''(x) = \frac{4}{(x - 2)^3} \)

a. (3 Points) Find the intercepts.

\[ 2x^2 - 3x = x(2x - 3) = 0 \]

\[ x = 0 \quad \text{or} \quad x = \frac{3}{2} \]

b. (5 Points) Find the asymptotes if any exist.

**Horizontal:**

\[ \lim_{x \to \infty} \frac{2x^2 - 3x}{x - 2} = \infty \]

\[ \lim_{x \to -\infty} \frac{2x^2 - 3x}{x - 2} = -\infty \]

\[ \{ \text{No HA} \} \]

**Vertical:**

\[ \lim_{x \to 2^+} \frac{2x^2 - 3x}{x - 2} = \infty \]

\[ x = 2 \quad \text{is} \quad \text{VA} \]

**Slant:**

\[ y = \frac{2x^2 - 3x}{x - 2} = 2x + 1 \]

\[ \text{Slo, slant asymptote} \]

\[ \frac{2x + 1}{x - 2} \]

c. (2 Points) Find the critical numbers.

\[ 2(x^2 - 4x + 3) = 2(x - 3)(x - 1) = 0 \]

\[ \therefore \text{Critical numbers:} \quad x = 1, \quad x = 3 \]

d. (4 Points) Find intervals where the function is increasing and those where it is decreasing.

Increasing on \((-\infty, 1) \cup (3, \infty)\)

Decreasing on \((1, 2) \cup (2, 3)\)

\[ f' \quad + \quad - \quad - \quad + \]

e. (2 Points) Find the local maximum and local minimum of the function.

at \(x = 1\) \ Local max. \ (1, 1) \]

at \(x = 3\) \ Local min. \ (3, 9) \]
f. (5 Points) Discuss the concavity of the function and find the infection points if any exist.

\[
\begin{array}{c}
f'' \quad - \quad + \\
\hline
2 \quad U
\end{array}
\]

Concave down on \((-\infty, 2)\)
Concave up on \((2, \infty)\)
No inflection point since 2 is not in the domain of the function.

g. (10 Points) Sketch the graph of the function. Clearly indicate the critical numbers, extrema and inflection points on the graph.