Problem 1 (17 points) A rectangle is constructed with its base on the diameter of a semicircle with radius 5 and its two other vertices on the semicircle. What are the dimensions of the rectangle with maximum area?

\[ A = 2xy = 2x \sqrt{25-x^2} \quad ; \quad 0 \leq x \leq 5 \]

\[ A' = 2 \frac{25-2x^2}{\sqrt{25-x^2}} + 2x \frac{-2x}{2 \sqrt{25-x^2}} \]

\[ x = \frac{5}{\sqrt{2}} \quad ; \quad y = \sqrt{25-\frac{25}{2}} = \frac{5}{\sqrt{2}} \]

\[ A \left( \frac{5}{\sqrt{2}} \right) = 25 \quad ; \quad A (0) = 0 = A (5) \]

Question 2: (35 points) Consider the function

\[ y = f(x) = \frac{x^3}{x^2-1} \quad \text{with} \quad f'(x) = \frac{x^2(x^2-3)}{(x^2-1)^2} \quad \text{and} \quad f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3} \]

a. (3 Points) Find the intercepts.

\[ x = 0 \quad , \quad y = 0 \]

b. (5 Points) Find the asymptotes if any exist.

Horizontal:

None \quad \text{because} \quad \lim_{x \to \pm\infty} \frac{x^3}{x^2-1} = \pm\infty

Vertical:

\[ x = 1 \quad \text{because} \quad \lim_{x \to 1^+} \frac{x^3}{x^2-1} = \infty \]

\[ x = -1 \quad \text{because} \quad \lim_{x \to -1^+} \frac{x^3}{x^2-1} = \infty \]

Slant:

None.

(1.73, 2.6)

\[ (0,0), \left( \sqrt{3}, \frac{3 \sqrt{3}}{2} \right), \left( -\sqrt{3}, -\frac{3 \sqrt{3}}{2} \right) \]

c. (2 Points) Find the critical numbers.

\[ x = 0 \quad x = \pm\sqrt{3} \]

d. (4 Points) Find intervals where the function is increasing and those where it is decreasing.

\[ f' : \begin{array}{cccccc}
+ & - & - & - & + \\
\sqrt{3} & -1 & 0 & 1 & \sqrt{3}
\end{array} \]

\[ \rightarrow \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \uparrow \]

\[ \text{at} \ x = -\sqrt{3} \quad \text{L. max} \quad \text{at} \ x = 0 \quad \text{None} \]

\[ \text{at} \ x = \sqrt{3} \quad \text{L. min} \]
f. (5 Points) Discuss the concavity of the function and find the inflection points if any exist.

Inflection points: \((0,0)\).  
\(f'' = - + - +\)

\(x = 0 \rightarrow y = 0\)

i.e \((0,0)\).

g. (10 Points) Sketch the graph of the function. Clearly indicate the critical numbers, extrema and inflection points on the graph.