Math201.10, Quiz #3, Term 171

Name:

ID #: Solutions **Serial #:**

1. [5 points] Find the local max/min values and saddle points of

$$f(x,y) = x^3 - 3x + 3xy^2.$$

2. [5 points] Find the extreme values of

$$f(x,y,z) = (x-1)^2 + (y+2)^2 + (z-1)^2$$

subject to the constraint $x^2 + y^2 + z^2 = 6$.

Good luck,

Ibrahim Al-Rasasi

[1] $f_x(x,y) = 3x^2 - 3 + 3y^2$; $f_y(x,y) = 6xy$

- f_x & f_y exist at all points in the xy-plane
- $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - 3 + 3y^2 = 0 & \sim (1) \\ 6xy = 0 & \sim (2) \end{cases}$ 1

$(2) \Rightarrow x = 0$ or $y = 0$

- $x = 0 \xRightarrow{(1)} -3 + 3y^2 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1 \Rightarrow \boxed{(0,-1), (0,1)}$
- $y = 0 \xRightarrow{(1)} 3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow \boxed{(-1,0), (1,0)}$ 1

- $f_{xx}(x,y) = 6x$; $f_{yy}(x,y) = 6x$; $f_{xy}(x,y) = 6y$ 1

$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - f_{xy}^2(x,y)$
$= (6x)(6x) - (6y)^2 = 36x^2 - 36y^2 = 36(x^2 - y^2)$

* $D(0,-1) = -36 < 0 \Rightarrow f$ has a saddle pt at $(0,-1)$

$D(0,1) = -36 < 0 \Rightarrow$ " " " " $(0,1)$

$D(-1,0) = 36 > 0$ & $f_{xx}(-1,0) = -6 < 0 \Rightarrow f$ has a local max at $(-1,0)$ 2

, the local max value is $f(-1,0) = 2$

$D(1,0) = 36 > 0$ & $f_{xx}(1,0) = 6 > 0 \Rightarrow f$ has a local min at $(1,0)$

, the local min value is $f(1,0) = -2$

2 $f(x,y,z) = (x-1)^2 + (y+2)^2 + (z-1)^2$; $g(x,y,z) = x^2+y^2+z^2-6$

- Solve the system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = 0 \end{cases} \Longrightarrow \begin{cases} 2(x-1) = \lambda(2x) \\ 2(y+2) = \lambda(2y) \\ 2(z-1) = \lambda(2z) \\ x^2+y^2+z^2-6=0 \end{cases} \Longrightarrow \begin{cases} x-1 = \lambda x & \text{---(1)} \\ y+2 = \lambda y & \text{---(2)} \\ z-1 = \lambda z & \text{---(3)} \\ x^2+y^2+z^2 = 6 & \text{---(4)} \end{cases}$$

1

- $(1) \Longrightarrow x - \lambda x = 1 \Longrightarrow x(1-\lambda) = 1 \Longrightarrow x = \dfrac{1}{1-\lambda}$ ---(5)

$(2) \Longrightarrow y - \lambda y = -2 \Longrightarrow y(1-\lambda) = -2 \Longrightarrow y = \dfrac{-2}{1-\lambda}$ ---(6)

$(3) \Longrightarrow z - \lambda z = 1 \Longrightarrow z(1-\lambda) = 1 \Longrightarrow z = \dfrac{1}{1-\lambda}$ ---(7)

1 $\lambda \neq 1$, otherwise $(1) \Longrightarrow x-1 = x \Longrightarrow -1 = 0$, impossible.

- Sub (5), (6), (7) in (4):

$$\frac{1}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 6$$

1

$$\Longrightarrow \frac{6}{(1-\lambda)^2} = 6 \Longrightarrow 6 = 6(1-\lambda)^2 \Longrightarrow (1-\lambda)^2 = 1 \Longrightarrow 1-\lambda = \pm 1 \Longrightarrow \lambda = 1 \pm 1 = \underline{\underline{0 \text{ or } 2}}$$

- $\lambda = 0 \xrightarrow{5,6,7} (x,y,z) = (1,-2,1)$

$\lambda = 2 \xrightarrow{5,6,7} (x,y,z) = (-1,2,-1)$ } 1

- $f(1,-2,1) = (1-1)^2 + (-2+2)^2 + (1-1)^2 = 0+0+0 = 0$

$f(-1,2,-1) = (-1-1)^2 + (2+2)^2 + (-1-1)^2 = 4+16+4 = 24$

- the $\underline{\underline{\text{max}}}$ value of f is 24 & it occurs at $(-1,2,-1)$

the $\underline{\underline{\text{min}}}$ value of f is 0 & ——— $(1,-2,1)$. } 1

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1. [5 points] Find the local max/min values and saddle points of

$$f(x,y) = x^2 + 2y^4 + 2xy.$$

2. [5 points] Find the extreme values of

$$f(x,y) = x^2 + 2y^2$$

subject to the constraint $x^2 + y^2 = 1$.

Good luck,

Ibrahim Al-Rasasi

$\boxed{1}$ $f_x(x,y) = 2x + 2y$; $f_y(x,y) = 8y^3 + 2x$

- f_x & f_y exist ~~at~~ all pts in the xy-plane

- $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 2x + 2y = 0 \\ 8y^3 + 2x = 0 \end{cases} \Rightarrow \begin{cases} x + y = 0 & \sim (1) \\ 4y^3 + x = 0 & \sim (2) \end{cases}$ 1

$(1) \Rightarrow x = -y \overset{(2)}{\Longrightarrow} 4y^3 - y = 0 \Rightarrow y(4y^2 - 1) = 0 \Rightarrow y(2y-1)(2y+1) = 0$

$\Rightarrow y = 0, \frac{1}{2}, -\frac{1}{2}$

- $y = 0 \Rightarrow x = 0 \Rightarrow \boxed{(0,0)}$
- $y = \frac{1}{2} \Rightarrow x = -\frac{1}{2} \Rightarrow \boxed{(\frac{1}{2}, -\frac{1}{2})}$ 1.5
- $y = -\frac{1}{2} \Rightarrow x = \frac{1}{2} \Rightarrow \boxed{(-\frac{1}{2}, \frac{1}{2})}$

- $f_{xx}(x,y) = 2$; $f_{yy}(x,y) = 24y^2$; $f_{xy}(x,y) = 2$

$D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - f_{xy}^2(x,y)$ 1

$= (2)(24y^2) - (2)^2 = 48y^2 - 4$

1.5

- $D(0,0) = -4 < 0 \Rightarrow f$ has a saddle pt at $(0,0)$
- $D(\frac{1}{2}, -\frac{1}{2}) = 12 - 4 = 8 > 0$ & $f_{xx}(\frac{1}{2}, -\frac{1}{2}) = 2 > 0 \Rightarrow f$ has a local min at $(\frac{1}{2}, -\frac{1}{2})$

  ∴ the local min value is $f(\frac{1}{2}, -\frac{1}{2}) = -\frac{1}{8}$

- $D(-\frac{1}{2}, \frac{1}{2}) = 12 - 4 = 8 > 0$ & $f_{xx}(-\frac{1}{2}, +\frac{1}{2}) = 2 > 0 \Rightarrow f$ has a local min at $(-\frac{1}{2}, \frac{1}{2})$

  ∴ The local min value is $f(-\frac{1}{2}, \frac{1}{2}) = -\frac{1}{8}$

4

[2] $f(x,y) = x^2 + 2y^2$; $g(x,y) = x^2 + y^2 - 1$

- Solve the System

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases} \Longrightarrow \begin{cases} 2x = \lambda(2x) \\ 4y = \lambda(2y) \\ x^2+y^2-1=0 \end{cases} \Longrightarrow \begin{cases} x = \lambda x & \text{---(1)} \\ 2y = \lambda y & \text{---(2)} \\ x^2+y^2=1 & \text{---(3)} \end{cases}$$ 1

- (1) $\Rightarrow x - \lambda x = 0 \Rightarrow x(1-\lambda) = 0 \Rightarrow \underline{x = 0 \text{ or } \lambda = 1}$ 1

  * $x = 0 \xrightarrow{(3)} 0 + y^2 = 1 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1 \Rightarrow \underline{(x,y) = (0,-1), (0,1)}$ 1
  * $\lambda = 1 \xrightarrow{(2)} 2y = y \Rightarrow y = 0 \xrightarrow{(3)} x^2 = 1 \Rightarrow x = \pm 1$

    $\Rightarrow \underline{(x,y) = (-1,0), (1,0)}$ 1

- $f(0,-1) = 0 + 2(-1)^2 = 2$

  $f(0,1) = 0 + 2(1)^2 = 2$

  $f(-1,0) = (-1)^2 + 2(0) = 1$

  $f(1,0) = 1^2 + 2(0) = 1$

- the max value of f is 2 & it occurs at $(0, \pm 1)$

  The min ——————— 1 ——————— $(\pm 1, 0)$ 1