King Fahd University of Petroleum & Minerals  
Department of Mathematics & Statistics  
Math 201  Major Exam I  
The First Semester of 2017-2018 (171)  
Time Allowed: 120 Minutes

- Mobiles, calculators and smart devices are not allowed in this exam.
- Provide all necessary steps required in the solution.
- Make sure that you have 10 different problems (10 pages + cover page). 

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Q:1 Consider the parametric equations of a curve $C$:
\[ x = \cos^2 \theta - \frac{1}{2}, \quad y = 4 \sin \theta \cos \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}. \]

(a) (7 points) Eliminate parameter $\theta$ to find the corresponding cartesian equation

\[ x = \cos^2 \theta - \frac{1}{2} \]
\[ = \frac{1}{2} (2 \cos^2 \theta - 1) \]
\[ = \frac{1}{2} \cos 2\theta \] \quad \text{(1)}

\[ y = 4 \sin \theta \cos \theta \]
\[ = 2 (2 \sin \theta \cos \theta) = 2 \sin 2\theta \] \quad \text{(2)}

(1) and (2) imply
\[ (2x)^2 + \left(\frac{y}{2}\right)^2 = 1 \]
\[ \Rightarrow \quad \frac{x^2}{\left(\frac{1}{2}\right)^2} + \frac{y^2}{2^2} = 1 \]

(b) (5 points) Sketch the curve and indicate with an arrow the direction in which the curve is traced as $\theta$ increases.
Q:2 Consider the parametric equations of a curve C:

\[ x = t^2 + 1, \quad y = e^t - 1. \]

(a) (5 points) Find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \).

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t}{2t}
\]

\[
\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) \cdot \frac{dx}{dt} = \frac{2te^t - 2e^t}{(2t)^2}
\]

\[
= \frac{2e^t(t-1)}{(2t)^3}
\]

(b) (3 points) For which values of \( t \) is the curve concave upward?

The curve is \( CV \), when \( \frac{d^2y}{dx^2} > 0 \)

\[ i.e \quad t < 0 \text{ or } t > 1 \]
Q:3 (8 points) At what point(s) on the curve \( x = 2t^3, \ y = 1 + 4t - t^2 \) does the tangent line have slope 2?

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4 - 2t}{6t^2}
\]

\[
2 = \frac{4 - 2t}{6t^2}
\]

\[
12t^2 + 2t - 4 = 0
\]

\[
6t^2 + t - 2 = 0
\]

\[
6t^2 + 4t - 3t + 2 = 0
\]

\[
3t(2t-1) + 2(2t-1) = 0
\]

\[
(3t+2)(2t-1) = 0
\]

\[
t = \frac{1}{2} \quad \text{or} \quad t = -\frac{2}{3}
\]

If \( t = \frac{1}{2} \), the point is \((\frac{1}{4}, \frac{11}{4})\)

and

If \( t = -\frac{2}{3} \), the point is \((-\frac{16}{27}, -\frac{19}{9})\).
Q: 4  (a) (4 points) Find the distance between the points with polar coordinates \((2, \frac{\pi}{3})\) and \((-4, -\frac{\pi}{3})\).

\[
\begin{align*}
(x_1, y_1) & = (2 \cos \frac{\pi}{3}, 2 \sin \frac{\pi}{3}) = (1, \sqrt{3}) \\
(x_2, y_2) & = (4 \cos \frac{2\pi}{3}, 4 \sin \frac{2\pi}{3}) = (-2, 2\sqrt{3}) \\
\text{Distance} & = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \\
& = \sqrt{(-3)^2 + (\sqrt{3})^2} \\
& = \sqrt{9 + 3} \\
& = \sqrt{12} = 2\sqrt{3}
\end{align*}
\]

(b) (5 points) Find the slope of the tangent line to the polar curve \(r = \frac{1}{\theta}\) at \(\theta = \pi\).

\[
\begin{align*}
\frac{dy}{dx} & = \frac{\frac{dr}{d\theta} \cdot \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cdot \cos \theta - r \sin \theta} \\
& = \frac{-\frac{1}{\theta^2} \cdot \sin \theta + \frac{1}{\theta} \cdot \cos \theta}{-\frac{1}{\theta^2} \cdot \cos \theta - \frac{1}{\theta} \cdot \sin \theta} \\
& = \frac{-\sin \theta + \theta \cdot \cos \theta}{-\cos \theta - \theta \cdot \sin \theta} \\
\left.\frac{dy}{dx}\right|_{\theta = \pi} & = \frac{-\sin \pi + \pi \cdot \cos \pi}{-\cos \pi - \pi \cdot \sin \pi} \\
& = \frac{-0 - \pi}{1 - 0} \\
& = -\pi
\end{align*}
\]
Q: 5(a) (4 points) Sketch the curve with the polar equation $r = 1 + \cos2\theta$ and indicate with an arrow the way it is traced as $\theta$ changes between $0$ and $2\pi$.

(b) (8 points) Calculate the arc length of the polar curve $r = \sin^3\left(\frac{\theta}{3}\right)$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

\[
L = \int_{0}^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta
= \int_{0}^{\pi/2} \sqrt{\sin^6\left(\frac{\theta}{3}\right) + \left(\sin^3\frac{\theta}{3} \cos\frac{\theta}{3}\right)^2} \, d\theta
= \int_{0}^{\pi/2} \sqrt{\sin^4\frac{\theta}{3} \left(\sin^2\frac{\theta}{3} + \cos^2\frac{\theta}{3}\right)} \, d\theta
= \int_{0}^{\pi/2} \sin^2\frac{\theta}{3} \, d\theta
= \int_{0}^{\pi/2} \frac{1}{2} \left(1 - \cos\frac{2\theta}{3}\right) \, d\theta
= \frac{1}{2} \left[ \theta - \frac{3}{2} \sin\frac{2\theta}{3} \right]_{0}^{\pi/2}
= \frac{1}{2} \left[ \frac{\pi}{2} - \frac{3}{2} \cdot \frac{\sqrt{3}}{2} \right] = \frac{2\pi - 3\sqrt{3}}{8}
\]
Q:6 (8 points) Find the area of the region enclosed by one loop of the polar curve 

\[ r = 3 \cos(3\theta). \]

\[ \gamma = 0 \Rightarrow 3 \cos(3\theta) = 0 \]
\[ \Rightarrow 3\theta = \pm \frac{\pi}{2} \]
\[ \Rightarrow \theta = \pm \frac{\pi}{6} \]

\[ A = \int_{0}^{\frac{\pi}{2}} \frac{1}{2} r^2 \, d\theta \]

\[ = 2 \int_{0}^{\frac{\pi}{6}} \frac{1}{2} (3\cos(3\theta))^2 \, d\theta \]
\[ = \int_{0}^{\frac{\pi}{6}} 9 \cos^2(3\theta) \, d\theta \]
\[ = \frac{9}{2} \int_{0}^{\frac{\pi}{6}} 1 + \cos(6\theta) \, d\theta \]
\[ = \frac{9}{2} \left[ \theta + \frac{\sin(6\theta)}{6} \right]_{0}^{\frac{\pi}{6}} \]
\[ = \frac{9}{2} \left[ \frac{\pi}{6} + 0 \right] = \frac{3\pi}{4} \]
Q:7 (a)(4 points) Find an equation of the largest sphere with center (2, 3, 4) that is contained in the first octant.

For the sphere to be contained in the first octant, we must have \( r = 2 \).

Equation:
\[
(x - 2)^2 + (y - 3)^2 + (z - 4)^2 = 2^2
\]

(b) (3 points) Find the highest point on the sphere in part (a).

Since \( r = 2 \), the required point is \((2, 3, 4 + 2)\)

\[
= (2, 3, 6)
\]
Q: 8 (a) (7 points) Find the unit vectors that are perpendicular to the tangent line to the curve \( y = 2 \sin x \) at \( \left( \frac{\pi}{6}, 1 \right) \).

The slope of the tangent line is
\[
\left. \frac{dy}{dx} \right|_{x=\frac{\pi}{6}} = 2 \cos x \bigg|_{x=\frac{\pi}{6}} = \sqrt{3}
\]
Slope of a line perpendicular to the tangent line is \(-\frac{1}{\sqrt{3}}\)
and a vector in this direction is \( \sqrt{3} \hat{i} - \hat{j} \).
Unit vectors are \( \pm \frac{1}{2} \left( \sqrt{3} \hat{i} - \hat{j} \right) \).

(b) (5 points) Find a vector \( \vec{u} \) in the direction of the vector from \( P(1, -2, 3) \) to \( Q(3, 1, -3) \) with length 3.

\[
\vec{PQ} = \langle 2, 3, -6 \rangle
\]
A unit vector in the direction of \( \vec{PQ} \) is
\[
\vec{u} = \frac{\vec{PQ}}{\|\vec{PQ}\|} = \left\langle \frac{2}{\sqrt{4+9+36}}, \frac{3}{\sqrt{4+9+36}}, -\frac{6}{\sqrt{4+9+36}} \right\rangle
\]
\[
= \langle \frac{2}{7}, \frac{3}{7}, -\frac{6}{7} \rangle
\]
The required vector is
\[
\vec{u} = 3 \vec{u} = \langle \frac{6}{7}, \frac{9}{7}, -\frac{18}{7} \rangle
\].
Q: 9(a) (6 points) If the angle between \( \vec{u} \) and \( \vec{v} \) is \( \theta = \frac{\pi}{6} \) and \( |\vec{u}| = 2, |\vec{v}| = \frac{1}{\sqrt{3}} \), then find \( |\vec{u} + \vec{v}| \).

\[
|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})
\]
\[
= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}
\]
\[
= |\vec{u}|^2 + 2 \vec{u} \cdot \vec{v} + |\vec{v}|^2
\]
\[
= 4 + 2|\vec{u}||\vec{v}| \cos \frac{\pi}{6} + \frac{1}{3}
\]
\[
= \frac{13}{3} + 2 \cdot 2 \cdot \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2}
\]
\[
= \frac{13}{2} + 2
\]
\[
= \frac{19}{3}
\]
\[
|\vec{u} + \vec{v}| = \sqrt{\frac{19}{3}}
\]

(b) (8 points) Let \( \vec{a} \) and \( \vec{b} \) be two unit vectors in the \( xyz \)-space. If the angle between them is \( \frac{2\pi}{3} \), then find \( (\vec{a} \times \vec{b}) \cdot (\vec{b} \times \vec{a}) \).

\[
\left( \vec{a} \times \vec{b} \right) \cdot \left( \vec{b} \times \vec{a} \right) = - \left( \vec{a} \times \vec{b} \right) \cdot \left( \vec{a} \times \vec{b} \right)
\]
\[
= - \left| \vec{a} \times \vec{b} \right|^2
\]
\[
= - \left( \left| \vec{a} \right| \left| \vec{b} \right| \sin \frac{2\pi}{3} \right)^2
\]
\[
= - \left( 1 \cdot 1 \cdot \sin \frac{2\pi}{3} \right)^2
\]
\[
= - \left( \frac{\sqrt{3}}{2} \right)^2
\]
\[
= - \frac{3}{4}
\]
Q:10 Let $P(-2, 1, 3), Q(4, 2, 5)$ and $R(1, 0, 1)$.

(a) (3 points) Find $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{QR}$

\[
\overrightarrow{PQ} = \langle 6, 1, 2 \rangle
\]
\[
\overrightarrow{PR} = \langle 3, -1, -2 \rangle
\]
\[
\overrightarrow{QR} = \langle -3, -2, -4 \rangle
\]

(b) (4 points) Find a non-zero vector orthogonal to the plane through the points $P, Q$ and $R$.

\[
\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
6 & 1 & 2 \\
3 & -1 & -2
\end{vmatrix}
\]
\[
= \langle 0, 18, -9 \rangle
\]

(c) (3 points) Find the area of the triangle $PQR$.

\[
A = \frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right|
\]
\[
= \frac{1}{2} \sqrt{0 + 18^2 + 9^2}
\]
\[
= \frac{1}{2} \sqrt{9^2 (2^2 + 1)}
\]
\[
= \frac{9}{2} \sqrt{5}
\]