King Fahd University of Petroleum & Minerals  
Department of Mathematics & Statistics  
Math 201  Major Exam II  
The First Semester of 2017-2018 (171)  
Time Allowed: 120 Minutes

Name: ___________________________  ID#: __________________
Section/Instructor: ___________________  Serial #: __________________

- Mobiles, calculators and smart devices are not allowed in this exam.
- Provide all necessary steps required in the solution.
- Make sure that you have 09 different problems (09 pages + cover page).

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Q:1(a) (06 points) Find the parametric equations of the line that passes through the points \( P(1, 3, 4) \) and \( Q(0, 5, 7) \).

A vector parallel to the line is \( \overrightarrow{PQ} = <-1, 2, 3> \).

Using point \( P(1, 3, 4) \), the parametric equations are

\[
\begin{align*}
&x = 1 - t \\
&y = 3 + 2t \\
&z = 4 + 3t
\end{align*}
\]

(b) (03 points) At what point does the line in part(a) intersect the \( xy- \) plane?

Line intersects \( xy- \) plane when \( z = 0 \), so we put \( z = 0 \) in line and obtain \( t = -\frac{4}{3} \).

Line intersects \( xy- \) plane at the point \( \left( \frac{7}{3}, \frac{1}{3}, 0 \right) \).

(c) (03 points) Where does the line in part(a) intersect the plane \( x - 2y - z + 1 = 0 \)?

The line and the plane intersect at

\[
\begin{align*}
(1-t) - 2(3+2t) - (4+3t) + 1 &= 0 \\
-1 - 4t - 3t + 1 - 6 - 4t + 1 &= 0 \\
-8t - 8 &= 0 \\
\Rightarrow t &= -1
\end{align*}
\]

Point of intersection is \( (2, 1, 1) \).
Q: 2 (10 points) Find the equation of the plane that passes through the points \((0, -2, 5)\) and \((-1, 3, 1)\) and is perpendicular to the plane \(2z = 5x + 4y\).

Let \(A = (0, -2, 5),\ B = (-1, 3, 1)\)

\[
\overrightarrow{AB} = \langle -1, 5, -4 \rangle
\]

\[
\vec{v} = \langle 5, 4, -2 \rangle
\]

Normal vector to the desired plane is \(\overrightarrow{AB} \times \vec{v}\)

\[
= \begin{vmatrix}
i & j & k \\
-1 & 5 & -4 \\
5 & 4 & -2 \\
\end{vmatrix}
\]

\[
= \langle 6, -22, -29 \rangle
\]

The equation of the plane is

\[
6(x - 0) - 22(y + 2) - 29(z - 5) = 0
\]

\[
6x - 22y - 29z = 44 - 145
\]

\[
6x - 22y - 29z = -101
\]
Q:3 (12 points) Consider the surface \( z = \sqrt{x^2 + 2y^2 - 4y + 2x + 3} \).

(a) Reduce the equation to one of the standard forms.

(b) Classify the surface (name, vertex, axis).

(c) Sketch the surface.

\[
\begin{align*}
    Z &= \sqrt{x^2 + 2y^2 - 4y + 2x + 3} \\
    Z^2 &= x^2 + 2x + 2y^2 - 4y + 3, \quad Z \geq 0 \\
    &= (x+1)^2 + 2(y-1)^2
\end{align*}
\]

(b) It is a half cone with vertex \((-1,1,0)\) and axis is parallel to the \(Z\)-axis.

(c)
Q:4 Let $f(x, y) = \sqrt{1-x^2+y^2}$.

(a) (08 points) Find and sketch the domain of $f$.

\[
\text{Domain} = \left\{ (x, y) \in \mathbb{R}^2 : 1-x^2+y^2 \geq 0 \right\} = \left\{ (x, y) \in \mathbb{R}^2 : x^2-y^2 \leq 1 \right\}
\]

(b) (06 points) Find an equation for the level curve of the function $f(x, y)$ that passes through the point $(1, 1)$.

Let $f(x, y) = R$. Then

\[
R = \sqrt{1+x^2-y^2} = \sqrt{1+1-1} = 1
\]

The equation for the level curve is

\[
\sqrt{1+x^2-y^2} = 1 \Rightarrow x^2-y^2 = 0.
\]
(a) (06 points) Find the limit, if it exists, or show that it does not exist:

\[
\lim_{(x,y) \to (0,0)} \frac{3xy \cos y}{x^2 + 4y^2}.
\]

Along the \( x \)-axis: \( \lim_{x \to 0} \frac{3x \cdot 0 \cdot \cos 0}{x^2 + 0} = 0 \)

Along the line \( y = x \):

\[
\lim_{y \to 0} \frac{3y^2 \cos y}{y^2 + 4y^2} = \frac{3}{5} \lim_{y \to 0} \cos y = \frac{3}{5}
\]

Limits are different. \( \text{DNE} \)

(b) (04 points) Determine the set of points at which the function \( f(x,y) = \frac{e^x + e^y}{e^{xy} - 1} \) is continuous.

\( e^x + e^y \) and \( e^y - 1 \) are continuous everywhere.

\( f(x,y) \) is continuous except where

\( e^{xy} - 1 = 0 \)

\( \Rightarrow xy = 0 \)

\( \Rightarrow x = 0 \) or \( y = 0 \).

Thus \( f(x,y) \) is continuous on domain

\[ \mathbb{D} = \{(x,y) : x \neq 0, y \neq 0\} \]
Let \( z = \frac{x^2 + y^2}{x + y} \).

(a) (02 points) Find \( \frac{\partial z}{\partial x} \).

\[
\frac{\partial z}{\partial x} = \frac{(x + y) \cdot 2x - (x^2 + y^2) \cdot 1}{(x + y)^2} = \frac{x^3 + 2xy - y^2}{(x + y)^2}
\]

(b) (02 points) Find \( \frac{\partial z}{\partial y} \).

\[
\frac{\partial z}{\partial y} = \frac{(x + y) \cdot 2y - (x^2 + y^2) \cdot 1}{(x + y)^2} = \frac{y^2 + 2xy - x^2}{(x + y)^2}
\]

(c) (04 points) Show that \( x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z \).

\[
x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{x^3 + 2x^2y - xy^2 + y^3 + 2xy^2 - yx^2}{(x + y)^2}
\]

\[
= \frac{x^3 + x^2y + y^3 + xy^2}{(x + y)^2}
\]

\[
= \frac{x^2(x + y) + y^2(y + x)}{(x + y)^2}
\]

\[
= \frac{(x + y)(x^2 + y^2)}{(x + y)^2}
\]

\[
= \frac{x^2 + y^2}{x + y}
\]

\[
= z
\]
Q:7 (12 points) Find the linear approximation of the function \( f(x, y) = \frac{xy \sin(x - y)}{1 + x^2 + y^2} \) at 
(1, 1) and use it to approximate \( f(1.02, 0.99) \).

**Solution:**

\[
f(x, y) = \frac{xy \sin(x - y)}{1 + x^2 + y^2}
\]

\[
f_x = \frac{(1 + x^2 + y^2)[y \sin(x - y) + xy \cos(x - y)] - xy \sin(x - y) \cdot 2x}{(1 + x^2 + y^2)^2}
\]

\[
f_x(1, 1) = \frac{1}{3}
\]

\[
f_y = \frac{(1 + x^2 + y^2)[x \sin(x - y) - xy \cos(x - y)] - xy \sin(x - y) \cdot 2y}{(1 + x^2 + y^2)^2}
\]

\[
f_y(1, 1) = -\frac{1}{3}
\]

\[
f(1, 1) = 0
\]

The linear approximation of the function \( f(x, y) \) at \((1, 1)\) is

\[
L(x, y) = f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1)
\]

\[
= 0 + \frac{1}{3}(x - 1) - \frac{1}{3}(y - 1)
\]

\[
= \frac{1}{3}(x - y)
\]

\[
f(1.02, 0.99) \approx \frac{1}{3}(1.02 - 0.99)
\]

\[
= \frac{1}{3}(0.03)
\]

\[
= 0.01
\]
Q:8 (08 points) Let \( z = f\left(\frac{r}{t}\right) \), where \( f \) is differentiable. Find \( r \frac{\partial z}{\partial r} + t \frac{\partial z}{\partial t} \) by the Chain rule. Simplify your answer.

Let \( x = \frac{x}{t} \). Then

\[ Z = f\left(\frac{x}{t}\right), \quad x = \frac{x}{t} \]

By the chain rule,

\[ \frac{\partial Z}{\partial y} = \frac{dz}{dx} \frac{\partial x}{\partial y} \]

\[ = f'(x) \frac{1}{t} \]

\[ \frac{\partial Z}{\partial x} = \frac{dz}{dx} \cdot \frac{\partial x}{\partial x} \]

\[ = f'(x) \left( -\frac{x}{t^2} \right) \]

Now,

\[ r \frac{\partial Z}{\partial y} + t \frac{\partial Z}{\partial x} = f'(x) \frac{x}{t} - f'(x) \frac{x}{t} \]

\[ = 0 \]
Q:9 (a) (08 points) Find the directional derivative of \( f(x, y, z) = xy^2z^3 \) at the point \( P(2, 1, 1) \) in the direction from \( P \) to \( Q(0, -3, 5) \).

\[
\vec{PQ} = \langle 0 - 2, -3 - 1, 5 - 1 \rangle = \langle -2, -4, 4 \rangle
\]

\[
\nabla f = \langle y^2z^3, 2xy^2z^3, 3xyz^2 \rangle
\]

\[
\nabla f(2, 1, 1) = \langle 1, 4, 6 \rangle
\]

unit vector \( \vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{1}{6} \langle -2, -4, 4 \rangle
\]

Therefore,

\[
D_{\vec{u}} f(2, 1, 1) = \nabla f(2, 1, 1) \cdot \vec{u}
\]

\[
= \frac{1}{6} \langle 1, 4, 6 \rangle \cdot \langle -2, -4, 4 \rangle
\]

\[
= \frac{1}{6} [-2 - 16 + 24] = 1
\]

(b)(06 points) Find all points at which the direction of fastest change of the function \( f(x, y) = x^2 + y^2 - 2x - 4y \) is \( i + j \).

\[
\nabla f(x, y) = \langle 2x - 2, 2y - 4 \rangle
\]

We need to find all points \( (x, y) \), where \( \nabla f(x, y) \) has the same direction as \( i + j \)

\[
\iff \langle 2x - 2, 2y - 4 \rangle = k \langle 1, 1 \rangle \quad (k > 0)
\]

\[
\iff 2x - 2 = k \quad \text{and} \quad 2y - 4 = k
\]

Then \( 2x - 2 = 2y - 4 \)

\[
\implies y = x + 1, \quad x > 1.
\]

Direction of fastest change is \( i + j \) at all points on the line \( y = x + 1, \quad x > 1 \).