King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics
Final EXAM - MATH 202
2017-2018 (171)

Monday, January 1, 2018 **Allowed Time: 3 Hours**

Name: ____

ID Number: ______________________ **Serial Number:** ______________

Section Number: _____________ **Instructor's Name:** ________________

Please read the following:

1. Exam has two parts: Part I: 5 Written Questions, Part II: 10 MCQs.
2. Provide all necessary steps with clear writing for Part I.
3. For Part II (MCQ), credit will be given only for the correct answer posted BELOW.
4. Mobiles and calculators are NOT allowed in this exam.

Part I: Written

Q #	Points	Max
1		15
2		14
3		18
4		10
5		13
Total Points		70

Part II: Multiple Choice [7 Points for each correct answer]

MCQ #	Student Answer	Points
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		
Total Points /70		

Grand Total/140	

Q1. (**15 points**) Find two linearly independent power series solutions of the differential equation

$$(x^2+1)y''+xy'-y=0$$

about the ordinary point $x_0=0$. Give the first four nonzero terms for each series solution.

Hint: The solution form $y=\sum_{n=0}^{\infty} c_n x^n$ and its first two derivatives lead to

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k+\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k+\sum_{k=1}^{\infty} kc_k x^k-\sum_{k=0}^{\infty} c_k x^k=0.$$

Solution: Above equation implies

$$2c_2-c_0+(6c_3+c_1-c_1)x+\sum_{k=2}^{\infty}[(k^2-1)c_k+(k+2)(k+1)c_{k+2}]\,x^k=0.$$

Comparing coefficients of powers of x gives

$$c_2=\frac{c_0}{2},$$

$$c_3=0,$$

$$c_{k+2}=-\frac{k-1}{k+2}c_k \quad \text{for } k\geq 2.$$

This recurrence relation generates the coefficients of the assumed solution

- $k=2 \Rightarrow \quad c_4=-\frac{1}{4}c_2=-\frac{1}{4}\left(\frac{1}{2}c_0\right)=-\frac{1}{8}c_0.$
- $k=3 \Rightarrow \quad c_5=-\frac{2}{5}c_3=0.$
- $k=4 \Rightarrow \quad c_6=-\frac{3}{6}c_4=-\frac{1}{2}(-\frac{1}{8}c_0)=\frac{1}{16}c_0.$

and so on. Now we substitute the coefficients just obtained into the original assumption

$$y=c_0y_1(x)+c_1y_2(x),$$

where

$$y_1(x)=1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6+\cdots,$$

$$y_2(x)=x.$$

Q2. (**14 points**) Use the method of Frobenius to find one series solution of the differential equation

$$x\,y'' + (1-x)y' - y = 0$$

about the regular singular point $x_0 = 0$. Then use the method of reduction of order to find the second linearly independent solution.

Hint: The solution form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ and its first two derivatives lead to

$$r^2 c_0 x^{-1} + \sum_{k=0}^{\infty}\left[(k+r+1)^2 c_{k+1} - (k+r+1)c_k\right] x^k = 0.$$

Solution: Comparing coefficients of powers of x gives

$$r^2 = 0,$$

$$c_{k+1} = \frac{1}{(k+r+1)} c_k \quad \text{for } k \geq 0.$$

For $r_1 = 0$ we get the recurrence relation $c_{k+1} = \frac{1}{(k+1)} c_k$ for $k \geq 0$ which generates the coefficients of the assumed solution

- $k = 0 \Rightarrow \quad c_1 = \frac{1}{1} c_0 = c_0.$
- $k = 1 \Rightarrow \quad c_2 = \frac{1}{2} c_1 = \frac{1}{2!} c_0.$
- $k = 2 \Rightarrow \quad c_3 = \frac{1}{3} c_2 = \frac{1}{3!} c_0.$
- $k = 3 \Rightarrow \quad c_4 = \frac{1}{4} c_3 = \frac{1}{4!} c_0.$

  ⋮
- $k = n \Rightarrow \quad c_n = \frac{1}{n!} c_0$ for $n \geq 1$.

Now we substitute the coefficients just obtained into the original assumption

$$y_1(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x .$$

- The second solution can be obtained directly as follow:

$$\begin{aligned} y_2(x) &= y_1(x)\int y_1^{-2}(x)\exp\left(-\int P(x)dx\right)dx \\ &= e^x \int e^{-2x}\exp\left(-\int \left(\frac{1}{x} - 1\right) dx\right) dx \\ &= e^x \int \frac{1}{x} e^{-x} dx \\ &= e^x \int \frac{1}{x}\left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots\right) dx \\ &= e^x \int \left(\frac{1}{x} - 1 + \frac{1}{2!}x - \frac{1}{3!}x^2 + \cdots\right) dx \\ &= e^x \ln x - e^x\left(x - \frac{1}{2(2!)}x^2 + \frac{1}{3(3!)}x^3 + \cdots\right) \\ &= e^x \ln x - e^x \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n!)} x^n \end{aligned}$$

Q3. (**18 points)** Solve the following initial-value problem

$$\mathbf{X}' = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}\mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} -2 \\ 8 \end{pmatrix}.$$

Solution:

We obtain the eigenvalue from

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 29 = 0.$$

The eigenvalues are $\lambda_1 = 5 + 2i$ and $\lambda_2 = \bar{\lambda}_1 = 5 - 2i$.

For λ_1 the corresponding eigenvector is obtained by solving $(\mathbf{A} - \lambda_1\mathbf{I}|\mathbf{0})$ and gives

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Now using $\mathbf{B}_1 = \mathrm{Re}(\mathbf{K}_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{B}_2 = \mathrm{Im}(\boldsymbol{K}_1) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$, $\alpha = \mathrm{Re}(\lambda_1) = 5$ and $\beta = \mathrm{Im}(\lambda_1) = 2$, the linearly independent solutions of the system are

$$\mathbf{X}_1 = [\mathbf{B}_1 \cos\beta t - \mathbf{B}_2 \sin\beta t]e^{\alpha t} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\sin 2t\right]e^{5t},$$

and

$$\mathbf{X}_2 = [\mathbf{B}_2 \cos\beta t + \mathbf{B}_1 \sin\beta t]e^{\alpha t} = \left[\begin{pmatrix} 0 \\ -2 \end{pmatrix}\cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin 2t\right]e^{5t}.$$

Hence, the general solution takes the form

$$\mathbf{X} = c_1\begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix}e^{5t} + c_2\begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix}e^{5t}.$$

Now the initial condition $\mathbf{X}(0) = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$ yields the algebraic system whose solution is

$$c_1 = -2, \qquad c_2 = -5.$$

Therefore, the solution is

$$\mathbf{X} = -2\begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix}e^{5t} - 5\begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix}e^{5t}.$$

Q4. (**10 points)** Consider the nonhomogeneous system

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 1/t \\ 1/t \end{pmatrix}.$$

If the general solution of the associated homogeneous system

$$\mathbf{X}_c = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} t+1 \\ t \end{pmatrix}.$$

Find a particular solution $\mathbf{X}_p$ of the nonhomogeneous system.

Solution: We note here that the entries in $\boldsymbol{X}_1$ form the first column of $\mathbf{\Phi}(t)$, and the entries in $\mathbf{X}_2$ form the second column of $\mathbf{\Phi}(t)$. Hence

$$\mathbf{\Phi}(t) = \begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1}(t) = \begin{pmatrix} -t & 1+t \\ 1 & -1 \end{pmatrix}.$$

Here $\mathbf{F}(t) = \begin{pmatrix} 1/t \\ 1/t \end{pmatrix}$.

Thus, the required particular solution is

$$\begin{aligned}
\mathbf{X}_p &= \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt \\
&= \begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix}\int \begin{pmatrix} -t & 1+t \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1/t \\ 1/t \end{pmatrix} dt \\
&= \begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix}\int \begin{pmatrix} 1/t \\ 0 \end{pmatrix} dt \\
&= \begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix}\begin{pmatrix} \ln t \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 1 \end{pmatrix}\ln t.
\end{aligned}$$

Q5. (**13 points)** Use the **matrix exponential method** to find the general solution of the following system

$$\mathbf{X}' = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix} \mathbf{X}.$$

Solution:

For $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}$, we have

$$\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix},$$

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the power series, we have

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2 t^2 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 3t & 0 & 0 \\ 5t & t & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3}{2}t^2 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ \frac{3}{2}t^2 + 5t & t & 1 \end{pmatrix}, \end{aligned}$$

Then we have

$$\begin{aligned} \mathbf{X} &= \mathbf{X}_c \\ &= e^{\mathbf{A}t}\mathbf{C} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ \frac{3}{2}t^2 + 5t & t & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 1 \\ 3t \\ \frac{3}{2}t^2 + 5t \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ t \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Q6. A lower bound for the radius of convergence of the power series solutions of the ODE

$(x^2 - 2x)y'' + x\,y' - 4y = 0$ about the ordinary point $x = 5$ is

(a) 3
(b) 1
(c) 2
(d) 4
(e) 5

Q7. Which of the following statements is TRUE about the ODE?

$$(x^2 - 16)y'' + (x + 4)y' + 2y = 0.$$

(a) $x = 4$ is a regular singular point
(b) $x = 4$ is an irregular singular point
(c) $x = -4$ is an irregular singular point
(d) $x = 4$ is an ordinary point
(e) $x = -4$ is an ordinary point

Q8. Given $x = 0$ is a regular singular point of the differential equation $x\,y'' + 3y' - 2y = 0$.

The sum of the indicial roots of the singularity is

(a) -2
(b) 2
(c) 0
(d) -4
(e) 4

Q9. The general solution of the ODE

$$\frac{dy}{dx} = (x + y + 3)^2$$

is

(a) $y = \tan(x + C) - x - 3$
(b) $y = \tan(x + C)$
(c) $y = \sin(x) - x - 4$
(d) $y = \sec(x + C) + x - 2$
(e) $y = \cot(x + C) - x - 1$

Q10. Which of the following statements is correct?

(a) Every Bernoulli ODE can be transformed into a linear first order ODE.
(b) Every homogenous ODE can be transformed into a linear first order ODE.
(c) Every Bernoulli ODE is homogenous.
(d) Every homogenous ODE is Bernoulli.
(e) Every exact ODE is Bernoulli.

Q11. If $a < b < c < d$ are the roots of the auxiliary equation of the ODE

$$y^{(4)} + 3y''' - 4y'' - 12y' = 0$$

then $a + 2b + c + d$ is equal to

(a) -5
(b) -1
(c) 2
(d) 5
(e) -2

Q12. The solution of the boundary-value problem $2x^2y'' + 3xy' - y = 0, \quad y(1) = 2, \; y(4) = \frac{9}{4}$,

satisfies y(9)=

(a) $\frac{28}{9}$
(b) $\frac{1}{9}$
(c) $\frac{1}{3}$
(d) $\frac{7}{9}$
(e) $\frac{25}{9}$

Q13. If $D^3 + a\,D^2 + b\,D + c$ annihilates $3e^{-2x} + \sin 2x$, then the value of $a + b + c$ is equal to

(a) 14
(b) 13
(c) -7
(d) -9
(e) -15

Q14. Given that $y_1(x) = x$ and $y_2(x) = x^3$ are two solutions of the homogeneous ODE

$x^2y'' - 3xy' + 3y = 0$, then the solution of the initial-value problem

$$x^2y'' - 3xy' + 3y = x, y(1) = 0, y'(1) = 0,$$

satisfies $y(e) =$

(a) $-\frac{3}{4}e + \frac{1}{4}e^3$
(b) e
(c) e^3
(d) $3e + \frac{1}{4}e^3$
(e) $-\frac{3}{4}e + 2\,e^3$

Q15. If we convert the Cauchy-Euler equation $2x^2y'' - xy' + 2y = 0$ into an equation with constant coefficients $y'' + a\,y' + b\,y = 0$, then $a + b$ is equal to

(a) $-\frac{1}{2}$
(b) $\frac{1}{2}$
(c) $-\frac{3}{2}$
(d) $-\frac{2}{3}$
(e) $-\frac{1}{3}$

Q#	Code 1	Code 2	Code 3	Code 4
6	b	c	d	e
7	d	e	a	b
8	c	d	e	a
9	a	b	c	d
10	e	a	b	c
11	d	c	b	a
12	e	d	c	b
13	c	b	a	e
14	e	d	b	c
15	a	e	c	d