King Fahd University of Petroleum & Minerals  
Department of Mathematics and Statistics  

MATH 302, Semester 171 (2017-2018)  

FINAL EXAM  
January 1, 2018  

Allowed Time: 180 mins  

Student Name:  
Student ID Number:  
Section Number:  

Instructions:  
1. Write neatly and legibly -- you may lose points for messy work.  
2. Show all your work -- no points for answers without justification.  
3. Programmable calculators and Mobiles are not allowed.  
4. Make sure that you have 10 problems and formula sheet (Total 12 pages).  

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Problem 1 (10 pts). Use elementary row operations (Gauss-Jordan) to find the inverse of the matrix

\[ A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix} \]
Problem 2 (18 pts). Find a matrix $P$ which diagonalizes

$$A = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -7 & 4 \\ -2 & 4 & -7 \end{pmatrix}$$
Problem 3 (12 pts). Let \( V \) be the region bounded by the two spheres
\[
x^2 + y^2 + z^2 = 4 \quad \text{and} \quad x^2 + y^2 + z^2 = 9.
\]
Find the outward flux \( \iint_S F \cdot dS \), if
\[
\vec{F} = \frac{\sin \theta}{r} \vec{a}_r + 3 \vec{a}_\theta + \sin \varphi \vec{a}_\varphi
\]
Problem 4 (20 pts).

a) For which values of $z = x + iy$, is the function

$$f(z) = \cos x - i\sin y$$

differentiable?

b) For what values $n$ is the function $u(x, y) = x^n - y^n$ harmonic?
Problem 5 (10 pts). Find all $z \in \mathbb{C}$ such that $\sin z = -\cos z$

Problem 6 (10 pts). Evaluate $\int_C (z^2 + z - 1)\,dz$,

where $C$ is the positively oriented quarter circle centered at 0 and of radius 2 in the first quadrant
Problem 7 (13 pts). Compute \[ \oint_{\gamma} \frac{\cos z}{z^3 + 9z} \, dz, \]

\( \gamma \): \(|z - 7| = 10 \) (positively oriented)
Problem 8 (12 pts). Find the Laurent Series of \( f(z) = \frac{1}{z(z-2)} \), for \( 0 < |z| < 2 \).
Problem 9 (20 pts). a) Let \( k \) be an integer. Show that \( \text{Res}(\cot(\pi z); k) = \frac{1}{\pi} \)

b) Suppose that a function \( f \) is analytic at an integer \( k \). Show that

\[
\text{Res}(f(z)\cot(\pi z); k) = \frac{1}{\pi} f(k)
\]

c) Evaluate \( \oint_C \frac{\cot(\pi z)}{1+z^4} \, dz \), \( C: |z - \frac{1}{4}| = \frac{1}{3} \) (positively oriented)
Problem 10 (15 pts). Compute the principal value of \[ \int_{-\infty}^{+\infty} \frac{2x^2 - 1}{x^4 + 5x^2 + 4} \, dx \]
Formulae in cylindrical and spherical coordinate systems

Differential of displacement

**Cylindrical:** \( dl = d\rho \, \hat{\rho} + \rho d\phi \, \hat{\phi} + dz \, \hat{z} \)

**Spherical:** \( dl = dr \, \hat{r} + r d\theta \, \hat{\theta} + r \sin \theta \, d\phi \, \hat{\phi} \)

Gradient of a scalar field, \( \nabla V \)

**Cylindrical:** 
\[
\nabla V = \frac{\partial V}{\partial \rho} \, \hat{\rho} + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \, \hat{\phi} + \frac{\partial V}{\partial z} \, \hat{z} = \left< \frac{\partial V}{\partial \rho}, \frac{1}{\rho} \frac{\partial V}{\partial \phi}, \frac{\partial V}{\partial z} \right>
\]

**Spherical:** 
\[
\nabla V = \frac{\partial V}{\partial r} \, \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \, \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \, \hat{\phi} = \left< \frac{\partial V}{\partial r}, \frac{1}{r} \frac{\partial V}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \right>
\]

Divergence of a vector field, \( \nabla \cdot G \)

**Cylindrical:** 
\[
\nabla \cdot G = \frac{1}{\rho} \frac{\partial (\rho G_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial (G_\phi)}{\partial \phi} + \frac{\partial G_z}{\partial z} = \left< \frac{\partial G_\rho}{\partial \rho}, \frac{1}{\rho} \frac{\partial G_\phi}{\partial \phi}, \frac{\partial G_z}{\partial z} \right>
\]

**Spherical:** 
\[
\nabla \cdot G = \frac{1}{r^2} \frac{\partial (r^2 G_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta G_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial G_\phi}{\partial \phi} = \left< \frac{\partial G_r}{\partial r}, \frac{1}{r \sin \theta} \frac{\partial G_\theta}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial G_\phi}{\partial \phi} \right>
\]

Relationship between Cartesian, Cylindrical and Spherical Coordinates

\[
\begin{pmatrix}
A_\rho \\
A_\phi \\
A_z
\end{pmatrix} = 
\begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix} 
\begin{pmatrix}
A_x \\
A_y \\
A_z
\end{pmatrix}
\]

\[
\begin{pmatrix}
A_r \\
A_\theta \\
A_\phi
\end{pmatrix} = 
\begin{pmatrix}
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\
\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0
\end{pmatrix} 
\begin{pmatrix}
A_x \\
A_y \\
A_z
\end{pmatrix}
\]

Differential of normal surface

**Cylindrical:** 
\( dS = \rho d\phi dz \, \hat{\rho} + d\rho dz \, \hat{\rho} + \rho d\rho d\phi \, \hat{\phi} \)

**Spherical:** 
\( dS = r^2 \sin \theta \, d\theta \, d\phi \, \hat{r} + \rho d\rho d\phi \, \hat{r} + r \sin \theta \, d\theta \, d\phi \, \hat{\phi} \)

Differential Volume

**Cylindrical:** 
\( dv = \rho d\rho d\phi dz \)

**Spherical:** 
\( dv = r^2 \sin \theta \, dr \, d\theta \, d\phi \)

Curl of a vector field, \( \nabla \times G \)

**Cylindrical:** 
\[
\nabla \times G = \frac{1}{\rho} \begin{vmatrix}
\hat{\rho} & \rho \hat{\phi} & \hat{z} \\
\frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\
G_\rho & \rho G_\phi & G_z
\end{vmatrix}, \quad \text{where } G = \left< G_\rho, G_\phi, G_z \right>
\]
Spherical:

\[ \nabla \times \mathbf{G} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{a}}_r & r \hat{\mathbf{a}}_\theta & r \sin \theta \hat{\mathbf{a}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ G_r & rG_\theta & r \sin \theta G_\phi \end{vmatrix}, \quad \text{where} \ G = \langle G_r, G_\theta, G_\phi \rangle \]

Laplacian of a scalar field, \( \nabla^2 V \)

Cylindrical: \( \nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \)

Spherical: \( \nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} \)

SCRATCH PAPER