King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 465 Final Exam
The First Semester of 2017-2018 (171)
Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

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<th>Problem #</th>
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Problem 1: Consider the system

\[ \frac{dx}{dt} = -x \quad (1) \]
\[ \frac{dy}{dt} = -2y - xy^2 \quad (2) \]

Sketch the trajectories of the system that pass through the point \((x, y) = (1, 1)\) for \(x > 0\) and \(x = 1\).

\[ \frac{dy}{dx} = \frac{2y + xy^2}{x} \]

\[ \int \frac{dy}{2y + xy^2} = \int \frac{dx}{x} \]

\[ \left[ \frac{1}{2y} - \frac{x}{x(2+xy)} \right] \frac{dy}{dx} = \ln|x| + C \]

\[ \frac{1}{2} \left[ \ln|y| - \ln|2+xy| \right] = \ln|x| + C \]

\[ y = \frac{C x^2}{2+xy} \]

\[ a) \quad x = 0, \quad y = C x^2 \]

\[ x = 1, y = 1 \Rightarrow C = 1 \]

\[ y = x^2, \quad x \in (0, 1) \]

\[ b) \quad x = 1, \quad y = \frac{2x^2}{1-Cx^2} \]

\[ x = 1, y = 1 \Rightarrow 1 = \frac{2C}{1-C} \]

\[ 1 \cdot C = \frac{1}{3} \]

\[ y = \frac{\sqrt{3} x^2}{1 - \frac{1}{3} x^2}, \quad x \in (\sqrt{3}, 1) \]
Problem 2: Consider the system

\[
\begin{align*}
\frac{dx}{dt} &= x^2 + y \\
\frac{dy}{dt} &= x + y^2.
\end{align*}
\]

(3) \hspace{8cm} (4)

Find and classify the critical points. Indicate their stability.

**Critical points**

\[
\begin{align*}
x^2 + y &= 0 \
\Rightarrow y &= -x^2 \\
x + y^2 &= 0 \
\Rightarrow x &= 0 \text{ or } x = -1
\end{align*}
\]

\(O(0, 0)\) and \(A(-1, -1)\)

At the origin:

System (3) - (4) is almost linear. Let consider the matrix \(A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\)

\[
\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0
\]

\(\lambda = 1\) or \(\lambda = -1\)

\(\Rightarrow \) the origin is a saddle unstable for the linear system \(\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = x \end{cases}\)

and for system (3)-(4)

At \(A(-1, 1)\)

let \(\begin{cases} z_1 = x + 1 \\ z_2 = y + 1 \end{cases}\)

\[
\begin{align*}
\frac{dz_1}{dt} &= z_1^2 - 2z_1 + z_2 \\
\frac{dz_2}{dt} &= z_2^2 - 2z_2 + z_1
\end{align*}
\]

(5) \hspace{8cm} (6)

let the matrix \(B \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\)

\[
\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (-2-\lambda)^2 - 1 = 0
\]

\(\lambda = -3\), \(\lambda = -1\)

\((0, 0)\) is a stable node for the linear system \(\frac{dz_1}{dt} = -2z_1 + z_2\)

\(\frac{dz_2}{dt} = -2z_2 + z_1\)

And (5) - (6) is almost linear system

\(\Rightarrow \) \((1, 1)\) is a stable node for (3)-(4).

Remark: Also you can use Jacobian matrix method.
**Problem 3**: Consider the system

\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= -y - 2x + 4x^3.
\end{align*}
\]  

Use the Lyapunov's second method to show that the zero solution \((x, y) = (0, 0)\) of (7)-(8) is asymptotically stable.

**Solution**

Let \(q(x) = 2x - 4x^3, \quad |x| \leq \frac{1}{8}\)

\[\Rightarrow \int_0^x q(x) \, dx = x - x^4, \quad |x| \leq \frac{1}{8}\]

\[q(x) \leq 4(x^2 - x^4), \quad |x| \leq \frac{1}{8}\]

\[\frac{dy}{dt} = -y - q(x) \times x^1\]

\[\frac{d}{dt} \left(\frac{y^2}{2} + x^2 - x^4\right) = -y^2\]

\[\beta \frac{d}{dt} (y g(x)) = \beta y g(x) + \beta y x^l g(x)\]

\[\frac{dV}{dt} = \frac{y^2}{2} + x^2 - x^4 + \beta y^2 g(x)\]

\[\Rightarrow \frac{d}{dt} V = -y^2 - \beta y g(x) - \beta y^2 g(x) + \beta y^2 g(x)\]

\[\Rightarrow \beta y^2 g(x) \leq \beta M y^2\]

\[\Rightarrow V \leq y^2 + \beta y g(x) - \beta y^2 g(x) - \beta y^2 g(x)\]

\[\Rightarrow -V \geq -\beta M y^2\]

We can choose \(\alpha, \beta, \gamma\) to make this hold.

\[\Rightarrow V \text{ is positive definite}
\]

and \(V^*\) is negative definite on \(D = \{(x, y) \mid |x| \leq \frac{1}{8}, \ y \leq \frac{1}{8}\}\)

\(\Rightarrow (0, 0)\) is asymptotically stable.
Problem 4: Consider the linear system

\[
\begin{align*}
\frac{dx}{dt} &= -x + 3y, \\
\frac{dy}{dt} &= -3x - y + y^2.
\end{align*}
\]

Estimate the region of asymptotic stability of the zero solution of the system.

\[
\text{Critical points:} \quad \begin{cases}
-x + 3y = 0 \\
-3x - y + y^2 = 0
\end{cases} \quad \Rightarrow \quad x = 3y, \quad y^2 - 10y = 0
\]

\[
\Rightarrow \quad (0, 0) \text{ and } (30, 10)
\]

At the origin:

\[
\begin{cases}
\frac{dx}{dt} = -x + 3y \\
\frac{dy}{dt} = -3x - y + y^2
\end{cases}
\]

\[
\frac{1}{2} \frac{d}{dt} x^2 = -x^2 + 3xy \\
\frac{1}{2} \frac{d}{dt} y^2 = -3xy - y^2 + y^3
\]

\[
\text{sum: } \quad \frac{d}{dt} (x^2 + y^2) = -x^2 - y^2 + y^3
\]

\[
V(x, y) = \frac{1}{2} (x^2 + y^2) \\
V^*(x, y) = -x^2 - y + y^3 = -[x^2 + y^2 (1 - y)]
\]

\[
D = \{ (x, y) \mid y < 1 \}
\]

\begin{itemize}
\item \( V \) is positive definite on \( D \).
\item \( V^* \) is positive definite on \( D \).
\end{itemize}

\[
C_x = \left\{ (x, y) \middle| V(x, y) \leq 1 \right\} = \left\{ (x, y) \middle| x^2 + y^2 \leq 2 \right\}
\]

The boundary of \( D \) is \( y = 1 \) and the point \( B(0) \) is on this boundary.

The curve \( V(x, y) = V(0, 1) = \frac{1}{2} \) meets the boundary.

\[
C_x = \left\{ (x, y) \middle| \frac{x^2 + y^2}{2} < \frac{1}{2} \right\} = \left\{ (x, y) \middle| x^2 + y^2 < 1 \right\}
\]

is included in the region of asymptotic stability.