1 (a) State Jordan Curve Theorem.

(b) Prove that $K_{3,3}$ is nonplanar.
(a) Show that $\alpha(C_5 \boxtimes C_5) = 5$.

(b) Show that a simple graph $G$ with all degrees at most $d$ satisfies

$$\alpha(G) \leq \frac{|V(G)|}{d + 1}.$$
3 (a) State Turán’s Theorem.

(b) Let $G$ be a simple graph with $m \geq \lfloor n^2/4 \rfloor + 1$. Show that $t(G) \geq \lfloor n/2 \rfloor$, where $t(G)$ is the number of triangles of $G$. 
4(a) An outerplanar graph is an undirected graph for which the vertices can be placed on a circle in such a way that no edges cross each other. Prove that any outerplanar graph is 3-vertex-colorable. A fact you may use without proof is that any outerplanar graph has a vertex of degree at most 2.

(b) For every integer \( t \geq 3 \), show that the Ramsey number \( r(3, t) \leq \frac{t^2 + 3}{2} \). (Hint: by induction on \( t \)).
(5) The probability space $\mathcal{G}_{(4,p)}$ has as sample space the $2^6 = 64$ spanning subgraphs of $K_4$ shown in the Figure below.

(a) Calculate the probability that a random graph $G$ in $\mathcal{G}_{(4,p)}$ is connected.

(b) Calculate the expectation $E(X)$ if $X$ denotes the number of components of $G \in \mathcal{G}_{(4,p)}$.

(c) Give an upper bound that is almost surely for the stability number of a random graph $G \in \mathcal{G}_{(4,p)}$. 


6 (a) Find an infinite family of graphs $G$ with $cr(G) = \frac{cm^3}{n^2}$, where $c$ is a suitable positive constant.

(b) Show that almost every random graph $G \in \mathcal{G}_{n,p}$ has diameter 2 and hence is connected.
7 (a) Let $G := (V, E)$ be a graph. Consider a random 2-colouring of $V$. Show that the expected number of edges of $G$ whose ends receive distinct colours is $m/2$.

(b) Deduce that every (loopless) graph $G$ contains a spanning bipartite subgraph $H$ with $e(H) \leq \frac{1}{2}e(G)$. 
EXTRA CREDITS (10 POINTS)
(8) Suppose we have a simple, undirected graph $G$ with $2n$ vertices and $2n$ edges, where $n \geq 3$. The graph consists of two disjoint cycles with $n$ edges each. For example, if $n = 6$, the graph would look like this:

A pair of vertices $u$ and $v$ from $G$ is selected uniformly at random from the pairs of distinct vertices with no edge between them. A new graph $G'$ is constructed to be the same as $G$, except that there is an edge between $u$ and $v$. What is the probability that $G'$ is connected?