Exercise 1. (5-5-5 points)
Let $V$ be the real vector space given by $V = \mathbb{R}^3$. Which one of the subsets of $V$ is a subspace of $V$. Justify.

1. $W_1 = \{ (x, y, z) \in V | x + y + z = 0 \}$.
2. $W_2 = \{ (x, y, z) \in V | x + y = 0 \text{ and } x - 2z = 1 \}$.
3. Find a complement of $W_i$ ($i = 1, 2$) when it is a subspace of $V$. 
Exercise 2. (6-6-8)

(1) Find explicitly a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $T(1, -1, 0) = (1, 0)$ and $T(-1, 0, 1) = (0, 1)$.

(2) Find the matrix representing $T$ in the standard bases $S_1, S_2$ of $\mathbb{R}^3$ and $\mathbb{R}^2$.

(3) Let $B_1 = \{(-1, 1, 0), (1, 0, 1), (0, 0, -1)\}$ and $B_2 = \{(1, -1), (-1, 2)\}$. Find the matrix representing $T$ in the bases $B_1$ and $B_2$. 
Exercise 3. (5-6-9 points)

Let $V$ be the real vector space given by $V = \mathbb{R}^3$ and $f_1, f_2, f_3$ the linear functionals of $V$ defined by: $f_1(x, y, z) = 2x + y + z$, $f_2(x, y, z) = x + 2y + z$, $f_3(x, y, z) = x + y + 2z$.

(1) Prove that \{f_1, f_2, f_3\} is a basis for $V^*$.

(2) Let $f$ be the linear functional of $V$ defined by $f(x, y, z) = x + y + z$. Express $f$ in the basis \{f_1, f_2, f_3\}.

(3) Find a basis $B$ of $V$ such that the dual basis $B^*$ of $B$ is the basis \{f_1, f_2, f_3\}. 
Exercise 4. (5-5-5)
Let $V$ be an $n$-dimensional vector space over a field $F$ and $W$ a subspace of $V$.

1. Prove that $\dim(W) + \dim(W^0) = \dim(V)$.

2. Assume that $V = M_2(\mathbb{R})$ the real vector space of all $2 \times 2$ matrices and $W = \{A \in V | AB = BA$ for all $B \in V\}$. Find explicitly $W$ and $\dim(W)$.

3. Show that $W^0$ is a hypersubspace of $V^*$. 
Exercise 5. (5-5-5-5)

Let $F$ be a field and $V$ be a finite dimensional space over $F$.

(1) Find all linear transformations of $F$ as a vector space over itself.

(2) Let $f$ a linear functional on $V$ such that $W = \ker f$ is a hypersubspace of $V$. Prove that for every linear functional $g$ of $V$ such that $g(W) = 0$, there is a scalar $c$ such that $g = cf$.

(3) Let $T$ be a linear operator on $V$ and $d$ a scalar in $F$ such that $Tu = du$ for some non-zero vector $u \in V$. Prove that there is a non-zero linear functional $h$ on $V$ such that $T^d(h) = dh$.

(4) Assume that $F = \mathbb{R}$, $V = \mathbb{R}^2$ and $T$ is defined by $T(a, b) = (2a, o)$. Find $u$, $d$ and $h$ satisfying question (3).